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81

Paul H. Rabinowitz
Edward W. Stredulinsky

Extensions of Moser–Bangert Theory

Locally Minimal Solutions

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Preface

This memoir is an outgrowth of earlier work of Moser and of Bangert on solutions of a family of nonlinear elliptic partial differential equations on $\mathbb{R}^n$ and of research of the authors on an Allen–Cahn PDE model of phase transitions. The simplest example of the class of equations studied by Moser and Bangert is

$$-\Delta u + F_u(x,u) = 0, \quad x \in \mathbb{R}^n, \tag{PDE}$$

where F is periodic in all of its arguments. Our earlier work was for equations of the form

$$-\Delta u + G_u(x,u) = 0, \quad x \in \mathbb{R}^2, \tag{AC}$$

where G is a double-well potential, e.g., $G(x,u) = a(x)u^2(1-u)^2$ with $a(x) > 0$ and 1-periodic in the components x_1, x_2 of x. The behaviors of F and G in u are rather different. However, the study of solutions of (AC) that lie between 0 and 1 can be reduced to a similar study for (PDE). Namely, taking G restricted to $\mathbb{R}^2 \times [0,1]$, extending it evenly and 2-periodically about $u = 0$, and rescaling the u variable yields an equation of the form of (PDE).

Moser initiated the study of a much more general family of equations than (PDE). His goal was to establish a version of the theory of Aubry and of Mather on monotone twist maps for partial differential equations. Toward that end, Moser and then Bangert studied solutions of their equations that possessed two additional properties: a certain minimality in a variational setting, and a so-called "without self intersections property" that will be explained later.

The goal of this monograph is to develop and study the rich structure of the set of solutions of the simpler model case (PDE), which both contains our earlier work on (AC) and expands the work of Moser and Bangert to include solutions that merely have local minimality properties. Minimization arguments are an important tool in our investigations. We begin in Part I by following Moser and using minimization arguments to obtain an ordered family of solutions of (PDE) that are 1-periodic in $x_1, \ldots, x_n$. Suppose there is a gap, i.e., no other members of this class, between a pair of such periodics. Then an ordered family of heteroclinic solutions in x_1 (and periodic in $x_2, \ldots, x_n$) between the pair are obtained by minimizing a "renormalized functional" associated with (PDE). Such basic heteroclinic solutions were originally obtained by Bangert. His argument was based on Moser's work and was not variational in nature. Our minimization approach is crucial for the construction of more complex solutions of (PDE) that, in the language of dynamical systems, shadow (or are near) formal concatenations of the basic heteroclinic states. These new multitransition solutions of (PDE) defined on $\mathbb{R} \times \mathbb{T}^{n-1}$ are studied in detail in Part II. They are obtained as local minima of the renormalized functional via a constrained minimization problem.

Whenever there is a gap between a pair of the basic heteroclinics in x_1, a second renormalized functional can be introduced and used to obtain ordered families of heteroclinic solutions in x_2 between them. The existence of such solutions by nonvariational arguments was also originally carried out by Bangert. The minimization approach to this new family of basic solutions of (PDE) is given in Part I. Lastly, it is used in Part III to construct further solutions of (PDE) defined on $\mathbb{R}^2 \times \mathbb{T}^{n-2}$ that shadow formal concatenations of the heteroclinics in x_2.

We thank Sergey Bolotin and Misha Feldman for many helpful conversations.

October, 2010

Paul H. Rabinowitz
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Contents

Chapter 1
Introduction

The goal of this memoir is to study the partial differential equation

$$-\Delta u + F_u(x, u) = 0, \quad x \in \mathbb{R}^n,$$

where F satisfies

$$F \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}) \tag{F$_1$}$$

and

$$F \text{ is } 1-\text{periodic in } x_1, \ldots, x_n \text{ and in } u. \tag{F$_2$}$$

Conditions (F_1)–(F_2) can be combined into the more concise condition

$$F \in C^2(\mathbb{T}^{n+1}, \mathbb{R}), \tag{F}$$

where $\mathbb{T}^{n+1} = \mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$.

The equation (PDE) is a special case of a much larger class of quasilinear elliptic partial differential equations studied by Moser [1] and by Bangert [2]. Seeking a codimension-1 analogue of results of Aubry [3] and of Mather [4] for monotone twist maps, Moser studied solutions u of (PDE) that were (i) minimal in the sense of Giaquinta and Giusti [5] and (ii) without self-intersections, or WSI for short. To explain (i)–(ii), set

$$L(u) = \frac{1}{2}|\nabla u|^2 + F(x, u),$$

the Lagrangian associated with (PDE), and

$$\mathcal{L}(u) = \int_{\mathbb{R}^n} L(u)dx.$$

Then calling u a minimal solution of (PDE) means

$$\mathcal{L}(u + \varphi) - \mathcal{L}(u) \geq 0 \tag{1.1}$$

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for all $\varphi \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ having compact support. Thus by (1.1), for any bounded domain $\Omega \subset \mathbb{R}^n$ with $\partial\Omega$ a smooth manifold, u minimizes $\mathcal{L}$ over the class of $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ functions v such that $v = u$ on $\mathbb{R}^n \setminus \Omega$. Condition (ii) means that for each $j \in \mathbb{Z}^n$ and $j_{n+1} \in \mathbb{Z}$, $u(x+j)-u(x)-j_{n+1}$ does not change sign on $\mathbb{R}^n$, i.e., there do not exist $y, z \in \mathbb{R}^n$ such that $y - z \equiv j \in \mathbb{Z}^n \setminus \{0\}$ and $u(y) - u(z) \equiv j_{n+1} \in \mathbb{Z}$ unless $u(x + j) = u(x) + j_{n+1}$ for all $x \in \mathbb{R}^n$.

Moser [1] and then Bangert [2] obtained a great deal of information about solutions of (PDE) that are minimal and WSI including the existence of elementary periodic solutions [1] and basic heteroclinic states that connect neighboring periodic solutions [2]. Our main goal in this memoir is to show that in addition to these solutions, there is an enormous number of additional more complex homoclinic and heteroclinic solutions of PDE that are "near", or in the language of dynamical systems, shadow formal concatenations of the basic states. These new solutions are not minimal and may not be WSI, but they possess a local minimality property. One of our motivations for seeking such solutions stems from an Allen–Cahn model of phase transitions that can be viewed as a very special case of (PDE) and for which these additional solutions represent possible phase-transition states.

To explain these statements, we begin by summarizing some of the work of Moser. In [1], he showed (in much greater generality):

Theorem 1.2. *If F satisfies (F_1)–(F_2) and u is a solution of (PDE) that is minimal and without self-intersections, there is an $\alpha = \alpha(u) \in \mathbb{R}^n$ such that*

$$|u(x) - \alpha \cdot x| \tag{1.3}$$

is bounded on $\mathbb{R}^n$.

The n-tuple α is called the rotation vector of the solution u. In the simplest case of $\alpha = 0$, u is bounded. Moser also proved:

Theorem 1.4. *For each $\beta \in \mathbb{R}^n$, there is a solution v of (PDE) that is minimal and without self-intersections such that $\alpha(v) = \beta$.*

For example, for $\alpha = 0$, such a v is obtained by minimizing

$$J_0(u) = \int_{\mathbb{T}^n} L(u)dx$$

over

$$\Gamma_0 = \{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \mid u \text{ is 1-periodic in } x_1, \ldots, x_n\}.$$

An analogous minimization argument produces v for $\alpha \in \mathbb{Q}^n$. See Chapter 5.

Suppose now that $\alpha = 0$. Except for Chapter 5, this is the case that we will study. Set

$$c_0 = \inf_{u \in \Gamma_0} J_0(u) \tag{1.5}$$

and

$$\mathcal{M}_0 = \{u \in \Gamma_0 \mid J_0(u) = c_0\}.$$

Moser further proved:

Theorem 1.6. *$\mathcal{M}_0$ is an ordered set, i.e., if $v, w \in \mathcal{M}_0$, then $v(x) \equiv w(x)$, $v(x) < w(x)$, or $v(x) > w(x)$ for all $x \in \mathbb{R}^n$.*

Since by (F_2), $u \in \mathcal{M}_0$ implies $u + j \in \mathcal{M}_0$ for any $j \in \mathbb{Z}$, Theorem 1.6 implies that either there is a continuum of members of $\mathcal{M}_0$ that join u and $u + 1$ and therefore $\mathcal{M}_0$ foliates $\mathbb{R}^{n+1}$ or there is a gap in $\mathcal{M}_0$ given by a pair of adjacent members $v_0, w_0 \in \mathcal{M}_0$ with $v_0 < w_0$. We will refer to v_0, w_0 as a *gap pair*. In the presence of such a gap pair, $\mathcal{M}_0$ merely laminates $\mathbb{R}^{n+1}$. Of course whenever there is one gap pair v_0, w_0, there are infinitely many, namely $v_0 + j$, $w_0 + j$ for any $j \in \mathbb{Z}$.

Assuming this gap condition, e.g. given by $v_0 < w_0$, Bangert [2] showed that there is a solution U_1 of (PDE) that is minimal and WSI and that is heteroclinic from v_0 to w_0 in x_1 and periodic in $x_2, \ldots, x_n$. Thus $U_1 \in C^2(\mathbb{R} \times \mathbb{T}^{n-1})$. Likewise there exists a solution $\overline{U}_1$ of (PDE) that is minimal and WSI and that is heteroclinic in x_1 from w_0 to v_0. For these results, x_1 can be replaced by x_i, $2 \leq i \leq n$, and even by any direction $j_1 e_1 + \cdots + j_n e_n$, where $e_1, \ldots, e_n$ is the usual Euclidean basis in $\mathbb{R}^n$ and $j \in \mathbb{Z}^n \setminus \{0\}$. The periodicity conditions in the remaining variables can also be generalized. See Chapter 5.

For $j \in \mathbb{Z}$ and $k \in \mathbb{N}$, set $\tau_j^k u(x) = u(x - je_k)$. Staying in the simplest setting, suppose U_1 is as above. Then $\tau_{-j}^1 U_1$ is also a solution of the same type, and Bangert further proved the set of such heteroclinic solutions is ordered. Thus

$$U_1 < \tau_{-1}^1 U_1. \tag{1.7}$$

More generally, when $u \leq \tau_{-1}^1 u$, we say that u is 1-*monotone in* x_1 and when (1.7) holds, we say that U is *strictly* 1-*monotone in* x_1. As above, either the region between U_1 and $\tau_{-1}^1 U_1$ in $\mathbb{R}^{n+1}$ is foliated by such solutions or there is a gap given by, e.g., an adjacent pair of solutions $v_1 < w_1$ lying between U_1 and $\tau_{-1}^1 U_1$. When such a gap is present, Bangert showed that there is a solution U_2 of (PDE) that is minimal and WSI and that is heteroclinic in x_2 from v_1 to w_1 and 1-periodic in $x_3, \ldots, x_n$, so $U_2 \in C^2(\mathbb{R}^2 \times \mathbb{T}^{n-2})$. Likewise there is a $\overline{U}_2$ heteroclinic in x_2 from w_1 to v_1. Continuing in this fashion with further observations about ordered sets of solutions and gap conditions, Bangert found more complicated heteroclinic solutions of (PDE) that were minimal and WSI.

Variants of what was just described for $\alpha = 0$ hold equally well for any $\alpha \in \mathbb{Q}^n$ and will be discussed in Chapter 5.

In the theory of dynamical systems, when one has families of basic solutions like $\{\tau_j^i U_i, \tau_k^i \overline{U}_i \mid j, k \in \mathbb{Z}\}$ for $i = 1, 2, \ldots, n$, one can often find further homoclinic and heteroclinic solutions of the associated equations that shadow phase shifts of the basic solutions, i.e., are near them in some sense. The simplest examples in our setting are solutions of (PDE) in $C^2(\mathbb{R} \times \mathbb{T}^{n-1})$ that are homoclinic to v_0 in x_1, near a phase shift of U_1 for large negative x_1 and near a phase shift for $\overline{U}_1$ for large positive x_1. Similarly, given U_2 and $\overline{U}_2$, one can seek solutions of (PDE) in $C^2(\mathbb{R}^2 \times \mathbb{T}^{n-2})$ that are homoclinic to v_1 in x_2, and have shadowing properties as above. Such solutions obtained by variational arguments have also been referred to as multibump solutions in the literature, but the terminology "multitransition solutions" seems more appropriate here. Thus the above homoclinics to v_0 are 2-transition solutions, and one can seek analogous $2k$-transition homoclinics to v_0 (or to w_0, v_1, or w_1) as well as $(2k+1)$-transition heteroclinics from v_0 to w_0, etc. The construction of such solutions is one of our main goals. In general these solutions will not be minimal and without self intersections although they will possess a local analogue of the minimality property (1.1).

Existence mechanisms to find such shadowing solutions have been developed in the settings of both dynamical systems and partial differential equations using constrained minimization arguments. See, e.g., Mather [6] for the dynamics case and [7, 8] for PDE results. For example, Mather [6] used such methods in his extensions of Aubry–Mather theory. There has also been a great deal of work using other variational approaches to find multitransition solutions for dynamical systems and partial differential equations. See, e.g., Séré [9, 10], who initiated work of this nature for dynamics problems and also Coti Zelati and Rabinowitz [11, 12]. The construction by Bangert [2] of the basic heteroclinic solutions that was mentioned above is not variational. Therefore before attempting to use minimization methods to find multitransition solutions of (PDE), a variational approach to obtain the basic heteroclinic solutions is needed. This is the main goal of the first part of this memoir. Work toward this end was initiated in [13,14] for the case of $\alpha = 0$ under the further hypothesis that

$$F \text{ is even in } x_1, \ldots, x_n. \tag{F_3}$$

This spatial reversibility condition yields functionals that are nonnegative and that can be analyzed much more simply than without (F_3).

Here we will drop (F_3) and in Chapter 5 also handle general $\alpha \in \mathbb{Q}^n$. In doing so, use will be made of some of the tools of [13, 14] and even more so of those developed in [7, 8], which considered an Allen–Cahn model of phase-transitions. In fact, our interest in (PDE) is an outgrowth of [7, 8] and earlier work on such phase- transition models by Alama, Bronsard, and Gui [15] and Alessio, Jeanjean, and Montecchiari [16, 17]. In [7, 8, 16, 17], model problems of the form

$$-\Delta u + G_u(x, u) = 0, \quad x \in \mathbb{R}^2, \tag{1.8}$$

were studied. Typically G is a double-well potential, e.g.,

$$G(x, u) = a(x)u^2(1-u)^2, \tag{1.9}$$

with $a(x) > 0$ and 1-periodic in x_1, x_2. Thus $u \equiv 0$ and $u \equiv 1$ are minima of G and solutions of (1.8). Of interest are further solutions of (1.8) that lie between 0 and 1 and are asymptotic to these basic states. Due to its definition in (1.9), G

is rather different from F in (PDE). However, if $G|_{\mathbb{R}^2\times[0,1]}$ is extended evenly in u to $\mathbb{R}^2 \times [-1,1]$ and then made 2-periodic in u, the resulting function $\widehat{G}$ satisfies $(F_1$–$F_2)$ (with period 2 in u rather than 1). For some other work on Allen–Cahn model equations, including results about cases not considered here such as nonperiodic dependence of F on x and irrational α, see, e.g., Alessio and Montecchiari [18–20], Bessi [21,22], and de la Llave and Valdinoci [23,24].

More generally, by their very nature, solutions of phase transition problems are heteroclinics or homoclinics for the associated differential equations. Thus the relatively simple equation (PDE) with its rich variety of solutions serves as a paradigm for the study of spatial phase-transition problems.

To outline what we will do here, beginning with the case of $\alpha = 0$, in Chapter 2 the function spaces and functional that will be used to find the basic heteroclinic solutions like U_1 and $\overline{U}_1$ will be introduced and their properties developed. Unfortunately, the natural functional to use to treat (PDE) in general is not bounded from below on any reasonable class of admissible functions. Therefore a new *renormalized functional* is introduced to overcome this difficulty. Then in Chapter 3, minimizing the renormalized functional establishes the existence of the basic heteroclinics. The relationship between the solutions of (PDE) obtained here by variational methods and those discovered by Bangert [2] will also be clarified. To obtain heteroclinics like U_2, $\overline{U}_2$ and their higher-dimensional counterparts in the most precise way, an induction argument should be employed. However, unlike the U_1 case, at the level of U_2 and higher, one has to deal with integrals over noncompact domains, and more technicalities are involved. Therefore the induction should begin after one has obtained U_2 and $\overline{U}_2$. The existence of U_2, $\overline{U}_2$ will be carried out in Chapter 4, mainly by indicating the changes needed in the framework of Chapters 2–3 to do so. Then Chapter 5 discusses how to modify the tools and constructions of the previous sections to extend the earlier results in three ways. In Section 5.1 higher-dimensional basic solutions defined on $\mathbb{R}^k \times \mathbb{T}^{n-k}$ are constructed. Then in Section 5.2 we find additional sets of basic solutions for the settings of Chapters 2–4 and Section 5.1. Finally in Section 5.3, the case of $\alpha \in \mathbb{Q}^n \backslash \{0\}$ in the contexts of Sections 5.1–5.2 and the earlier chapters is treated.

Parts II and III of the memoir employ the basic solutions of Part I to construct shadowing or multitransition solutions on $\mathbb{R} \times \mathbb{T}^{n-1}$ and on $\mathbb{R}^2 \times \mathbb{T}^{n-2}$ respectively. Several comparison results that will be useful for this purpose are obtained in Chapter 6 and then used in Chapter 7 to establish the existence of infinitely many two-transition solutions defined on $\mathbb{R} \times \mathbb{T}^{n-1}$, lying between v_0 and w_0, homoclinic to v_0, and shadowing phase shifts of U_1 and $\overline{U}_1$. In Chapter 8, we extend the results of Chapters 6–7 on 2-transition homoclinic solutions of (PDE) in the gap between v_0 and w_0 to k- and ∞-transition homoclinic and heteroclinic solutions. While this can be done by a direct generalization of the methods used for the simpler case, we introduce another more geometrical construction in the spirit of [7, 8]. Chapters 9–10 study 2-transition solutions of (PDE) which are strictly 1-monotone in x_1 (in the sense of (1.7)). Assuming that we have an ordered pair of gap pairs $v_0 < w_0 \leq \hat{v} < \hat{w}$, in Chapter 9 the existence of infinitely many heteroclinics strictly 1-monotone in x_1 from v_0 to $\hat{w}$ is established. Such solutions are present even if the region between w_0 and $\hat{v}$ contains continua of periodics or of heteroclinics. Several technical results that are required in Chapter 13 are also proved in Chapter 9. Then Chapter 10 shows how to extend the results of Chapter 9 to find k-transition solutions of (PDE) that are strictly1-monotone in x_1. Part II concludes with Chapter 11 where a study is made of solutions with behavior intermediate to those of Chapters 6–10. Thus we treat cases in which there are multitransition solutions of (PDE) on $\mathbb{R} \times \mathbb{T}^{n-1}$ that neither are 1-monotone in x_1 nor lie in a gap in $\mathcal{M}_0$. This requires ideas from the regularity theory of variational inequalities, and we are indebted to Misha Feldman for his essential contributions here.

There are natural analogues of the results of Chapters 6–11 in the context of solutions on $\mathbb{R}^2 \times \mathbb{T}^{n-2}$. However, we do not pursue them in Part III, but rather study two cases that involve new phenomena. In Chapter 12, our main interest is in the construction of solutions of (PDE) that are in a sense concatenations in x_2 of an infinite number of phase shifts of U_2, are strictly 1-monotone in x_1 and x_2, and are heteroclinic from v_0 to w_0 in both x_1 and x_2. This involves in part a monotone rearrangement argument that is of independent interest. Then lastly, in Chapter 13, we again study the existence of solutions of (PDE) that are strictly 1-monotone in x_1 and x_2 but now are heteroclinic in x_2 between a pair of solutions of (PDE) that are 1-monotone in x_1 and are of the type obtained in Chapter 9. This final case is the most technically demanding one that we treat.

Part I
Basic Solutions

Chapter 2
Function Spaces and the First Renormalized Functional

Suppose $\mathcal{M}_0$ is as in the introduction, and the gap condition

$$\text{there are adjacent } v_0, w_0 \in \mathcal{M}_0 \text{ with } v_0 < w_0 \tag{*$_0$}$$

holds. Our goal is to show there are solutions of (PDE) heteroclinic in x_1 from v_0 to w_0 and 1-periodic in the remaining variables. This requires introducing a class of admissible functions and an appropriate functional on this class whose minima will be the desired solutions of (PDE). As a first attempt, take the class of $W^{1,2}_{loc}(\mathbb{R}\times\mathbb{T}^{n-1})$ functions that are asymptotic to v_0 and w_0 as $x_1 \to \pm\infty$ in some reasonable way and

$$\text{minimize } \int_{\mathbb{R}\times\mathbb{T}^{n-1}} L(u)dx \tag{2.1}$$

over this class. By writing $u \in W^{1,2}_{loc}(\mathbb{R}\times\mathbb{T}^{n-1})$, we mean that $u(x+e_i) = u(x)$, $2 \le i \le n$. Unfortunately, this functional may not be bounded from below. In addition, if $F > 0$ on $\mathbb{T}^{n+1}$, the functional will be infinite for any admissible u. Thus a more careful approach is required, and the functional in (2.1) must be modified. Such a "renormalized" functional that is bounded from below will be introduced. Toward that end, let $v, w \in \mathcal{M}_0$, $v < w$, and define

$$\widehat{\Gamma}_1 = \widehat{\Gamma}_1(v,w) = \{u \in W^{1,2}_{loc}(\mathbb{R}\times\mathbb{T}^{n-1}) \mid v \le u \le w\}.$$

For $i \in \mathbb{Z}$, set $T_i = [i, i+1]\times\mathbb{T}^{n-1}$. Now for $u \in \widehat{\Gamma}_1$ and $i \in \mathbb{Z}$, define

$$J_{1,i}(u) = \int_{T_i} L(u)dx - c_0$$

with c_0 as in (1.5). For $p, q \in \mathbb{Z}$ with $p \le q$ and $u \in \widehat{\Gamma}_1$, set

$$J_{1;p,q}(u) = \sum_{i=p}^{q} J_{1,i}(u).$$

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It is easily seen that $J_{1;p,q}(u)$ is bounded from below, but its lower bound may depend on $q-p$. The next result helps us obtain a better lower bound.

Proposition 2.2. *Let* $\ell \in \mathbb{N}^n$ *and*

$$\Gamma_0(\ell) = \{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \mid u(x+\ell_i e_i) = u(x), 1 \le i \le n\}.$$

Set

$$J_0^\ell(u) = \int_0^{\ell_1} \cdots \int_0^{\ell_n} L(u)dx$$

and

$$c_0(\ell) = \inf_{u \in \Gamma_0(\ell)} J_0^\ell(u). \tag{2.3}$$

Then

$$\mathcal{M}_0(\ell) = \{u \in \Gamma_0(\ell) \mid J_0^\ell(u) = c_0(\ell)\} \neq \emptyset.$$

Moreover, $\mathcal{M}_0(\ell) = \mathcal{M}_0$ *and* $c_0(\ell) = (\prod_1^n \ell_i)c_0$.

Proof. The proof of Proposition 2.2 is contained in Moser's work [1]. Some of the arguments will be required repeatedly in this paper, so it is convenient to give the proof in the current simple setting. Since J_0^ℓ is weakly lower semicontinuous on $\Gamma_0(\ell)$, $\mathcal{M}_0(\ell) \neq \emptyset$. Moreover, standard elliptic regularity arguments show that $u \in \mathcal{M}_0(\ell)$ implies that u is a classical solution of (PDE).

Next it will be shown that $\mathcal{M}_0(\ell)$ is an ordered set. If not, there exist $v, w \in \mathcal{M}_0(\ell)$ and $\xi, \eta \in \prod_{i=1}^n(\ell_i \mathbb{T}^1)$ such that $v(\xi) = w(\xi)$ and $v(\eta) < w(\eta)$. Set $\varphi = \max(v,w)$ and $\psi = \min(v,w)$. Then $\varphi, \psi \in \Gamma_0(\ell)$ and

$$J_0^\ell(\varphi) + J_0^\ell(\psi) = J_0^\ell(v) + J_0^\ell(w) = 2c_0(\ell). \tag{2.4}$$

Since

$$J_0^\ell(\varphi), J_0^\ell(\psi) \ge c_0(\ell),$$

(2.4) implies $J_0^\ell(\varphi) = J_0^\ell(\psi) = c_0(\ell)$, so $\varphi, \psi \in \mathcal{M}_0(\ell)$. Therefore φ and ψ are classical solutions of (PDE) with $\varphi \ge \psi$, $\varphi(\xi) = \psi(\xi)$, and $\varphi(\eta) > \psi(\eta)$. Thus $f \equiv \varphi - \psi \ge 0$ and satisfies the linear elliptic partial differential equation

$$-\Delta f + af = -bf, \quad x \in \mathbb{R}^n, \tag{2.5}$$

where $a = \max(A,0)$, $b = \min(A,0)$, and

$$A = \begin{cases} \dfrac{F_u(x,\varphi(x)) - F_u(x,\psi(x))}{\varphi(x)-\psi(x)} & \text{if } \varphi(x) > \psi(x), \\ F_{uu}(x,\varphi(x)) & \text{if } \varphi(x) = \psi(x). \end{cases}$$

Since $a \geq 0$ and $b \leq 0$ are continuous, the elliptic maximum principle applies to (2.5) and shows that either $f \equiv 0$ or $f > 0$ in $\mathbb{R}^n$. But $f(\xi) = 0$ and $f(\eta) > 0$, a contradiction. Hence no such v and w exist and $\mathcal{M}_0(\ell)$ is an ordered set.

Now to prove the final assertions of Proposition 2.2, let $u \in \mathcal{M}_0(\ell)$. If each $u \in \mathcal{M}_0(\ell)$ satisfies

$$u(x + e_i) = u(x), \quad 1 \leq i \leq n, \tag{2.6}$$

then $\mathcal{M}_0(\ell) = \mathcal{M}_0$ and $c_0(\ell) = (\prod_{i=1}^{n} \ell_i)c_0$. To verify (2.6), suppose $u \in \mathcal{M}_0(\ell)$. Since $u(x + e_i) \in \mathcal{M}_0(\ell)$, $i = 1, \ldots, n$, and $\mathcal{M}_0(\ell)$ is ordered, either (2.6) holds or

$$\text{(i) } u(x + e_i) > u(x) \quad \text{or} \quad \text{(ii) } u(x + e_i) < u(x) \tag{2.7}$$

for each i. But if (2.7) (i) is satisfied,

$$u(x) = u(x + \ell_i e_i) \geq \cdots \geq u(x + e_i) > u(x),$$

a contradiction. A similar argument when (2.7) (ii) holds shows that (2.6) is valid, and the proposition is proved.

Now a better lower bound for $J_{1;p,q}(u)$ can be obtained.

Proposition 2.8. *There is a constant $K_1 \geq 0$, depending on v and w but independent of $p, q \in \mathbb{Z}$ and $u \in \widehat{\Gamma}_1$, such that*

$$J_{1;p,q}(u) \geq -K_1.$$

Proof. Let $u \in \widehat{\Gamma}_1$. Then

$$\begin{aligned} J_{1,p}(u) = &\int_{T_p} \Big(\frac{1}{2}|\nabla(u - v)|^2 + \nabla(u - v) \cdot \nabla v \\ &\qquad + \frac{1}{2}|\nabla v|^2 + F(x, u) - F(x, v) + F(x, v) \Big)\, dx - c_0 \\ = &\ \frac{1}{2}\|\nabla(u - v)\|^2_{L^2(T_p)} + \int_{T_p} (\nabla(u - v) \cdot \nabla v + F(x, u) - F(x, v)) dx. \end{aligned} \tag{2.9}$$

Now

$$\left| \int_{T_p} (F(x, u) - F(x, v))\, dx \right| \leq M_1 \|w - v\|_{L^\infty(T_0)}, \tag{2.10}$$

where $M_1 = \max_{\mathbb{T}^{n+1}} |F_u(x, u)|$. Also

$$\int_{T_p} \nabla(u - v) \cdot \nabla v\, dx = \int_{\partial T_p} (u - v) \frac{\partial v}{\partial \nu} dS - \int_{T_p} (u - v) \Delta v\, dx, \tag{2.11}$$

where ν denotes the outward-pointing normal. Since $u \in \widehat{\Gamma}_1$ and v is a solution of (PDE),

$$\left|\int_{T_p}(u-v)\Delta v\,dx\right| \le \|F_u(\cdot,v)\|_{L^\infty(T_p)}\int_{T_p}(w-v)\,dx \le M_1\|w-v\|_{L^\infty(T_0)}. \tag{2.12}$$

The boundary term in (2.11) can be estimated by

$$\left|\int_{\partial T_p}(u-v)\frac{\partial v}{\partial \nu}dS\right| \le 2\left\|\frac{\partial v}{\partial x_1}\right\|_{L^\infty(T_0)}\|w-v\|_{L^\infty(T_0)}. \tag{2.13}$$

Combining (2.9)–(2.13) yields

$$\left|J_{1,p}(u)-\frac{1}{2}\|\nabla(u-v)\|^2_{L^2(T_p)}\right| \le M_2\|w-v\|_{L^\infty(T_0)}, \tag{2.14}$$

where $M_2 = 2M_1 + 2\|\frac{\partial v}{\partial x_1}\|_{L^\infty(T_0)}$. This proves Proposition 2.8 for $q = p$, $p+1$, or $p+2$ with $K_1 = 3M_2\|w-v\|_{L^\infty(T_0)}$. Thus suppose that $q > p+2$. Define χ via

$$\chi = \begin{cases} v, & x_1 \le p, \\ (x_1-p)u + (p+1-x_1)v, & p \le x_1 \le p+1, \\ u, & p+1 \le x_1 \le q, \\ (x_1-q)v + (q+1-x_1)u, & q \le x_1 \le q+1, \\ v, & q+1 \le x_1. \end{cases} \tag{2.15}$$

Then χ extends naturally to $\widehat{\Gamma}_1$ as a $(q+1-p)$-periodic function of x_1. Hence by Proposition 2.2,

$$0 \le J_{1;p,q}(\chi) = J_{1,p}(\chi) + J_{1;p+1,q-1}(u) + J_{1,q}(\chi),$$

or

$$J_{1;p+1,q-1}(u) \ge -J_{1,p}(\chi) - J_{1,q}(\chi). \tag{2.16}$$

Next observe that

$$\chi - v = (x_1-p)(u-v), \quad p \le x_1 \le p+1,$$

so

$$\begin{aligned} |\nabla(\chi-v)|^2 &= (x_1-p)^2|\nabla(u-v)|^2 + (u-v)^2 + 2(x_1-p)(u-v)\frac{\partial}{\partial x_1}(u-v) \\ &= (x_1-p)^2|\nabla(u-v)|^2 + \frac{\partial}{\partial x_1}((x_1-p)(u-v)^2) \end{aligned}$$

and

$$\|\nabla(\chi - v)\|^2_{L^2(T_p)} \le \|\nabla(u - v)\|^2_{L^2(T_p)} + \|w - v\|^2_{L^\infty(T_0)}. \tag{2.17}$$

Hence by (2.14) and (2.17),

$$J_{1,p}(\chi) \le \frac{1}{2}\|\nabla(u - v)\|^2_{L^2(T_p)} + M_2\|w - v\|_{L^\infty(T_0)} + \frac{1}{2}\|w - v\|^2_{L^\infty(T_0)}. \tag{2.18}$$

Finally,

$$\begin{aligned} J_{1;p,q}(u) &= J_{1,p}(u) + J_{1;p+1,q-1}(u) + J_{1,q}(u) \\ &\ge -4M_2\|w - v\|_{L^\infty(T_0)} - \|w - v\|^2_{L^\infty(T_0)} \\ &\equiv -K_1. \end{aligned} \tag{2.19}$$

Remark 2.20. If $v = v_0$ and $w = w_0$, $\|w - v\|_{L^\infty(T_0)} \le 1$.

The lower bound for $J_{1;p,q}(u)$ provided by Proposition 2.8 suggests defining

$$J_1(u) = \varliminf_{\substack{p \to -\infty \\ q \to \infty}} J_{1;p,q}(u) \tag{2.21}$$

for $u \in \widehat{\Gamma}_1$. For J_1 so defined, there is also an upper bound for $J_{1;p,q}(u)$:

Lemma 2.22. *If $u \in \widehat{\Gamma}_1$ and $p, q \in \mathbb{Z}$ with $p \le q$,*

$$J_{1;p,q}(u) \le J_1(u) + 2K_1. \tag{2.23}$$

Proof. By (2.21) and Proposition 2.8,

$$\begin{aligned} J_1(u) &= \varliminf_{s \to -\infty} J_{1;s,p-1}(u) + J_{1;p,q}(u) + \varliminf_{t \to \infty} J_{1;q+1,t}(u) \\ &\ge J_{1;p,q}(u) - 2K_1. \end{aligned}$$

Define

$$\begin{aligned} \Gamma_1 \equiv \Gamma_1(v, w) \equiv \big\{ u \in \widehat{\Gamma}_1 \mid \|u - v\|_{L^2(T_i)} \to 0, i \to -\infty, \\ \|u - w\|_{L^2(T_i)} \to 0, i \to \infty \big\}. \end{aligned}$$

Fortunately, the expression for J_1 simplifies when we are dealing with $u \in \Gamma_1$, since the $\underline{\lim}$'s in (2.21) become limits. The next result shows this and more:

Proposition 2.24. *If $u \in \Gamma_1$ and $J_1(u) < \infty$, then*

$$J_{1,i}(u) \to 0, \quad |i| \to \infty, \tag{2.25}$$

$$\|\tau^1_{-i} u - v\|_{W^{1,2}(T_0)} \to 0, \quad i \to -\infty, \tag{2.26}$$

$$\|\tau^1_{-i} u - w\|_{W^{1,2}(T_0)} \to 0, \quad i \to \infty, \tag{2.27}$$

$$J_1(u) = \lim_{\substack{p \to -\infty \\ q \to \infty}} J_{1;p,q}(u). \tag{2.28}$$

Proof. By (2.23) with $p = q = i$, $J_{1,i}(u)$ is bounded from above independently of $i \in \mathbb{Z}$. Hence by (2.14), $\|\nabla(\tau^1_{-i} u - v)\|_{L^2(T_0)}$ is bounded independently of $i \in \mathbb{Z}$. Since

$$\|\tau^1_{-i} u - v\|_{L^2(T_0)} \le \|w - v\|_{L^\infty(T_0)}, \tag{2.29}$$

$\tau^1_{-i} u - v$ is bounded in $W^{1,2}(T_0)$. Therefore there is a $\varphi \in W^{1,2}(T_0)$ such that $\tau^1_{-i} u - v \to \varphi$ weakly in $W^{1,2}(T_0)$ and strongly in $L^2(T_0)$ for a subsequence of i's $\to -\infty$. But since $u \in \Gamma_1$, $\|\tau^1_{-i} u - v\|_{L^2(T_0)} \to 0$ as $i \to -\infty$. Hence $\varphi = 0$, and it readily follows that $\tau^1_{-i} u \to v$ weakly in $W^{1,2}(T_0)$ and strongly in $L^2(T_0)$ as $i \to -\infty$ along the full sequence. By the weak convergence in $W^{1,2}(T_0)$,

$$\int_{T_0} \nabla v \cdot \nabla(\tau^1_{-i} u - v)\,dx \to 0, \quad i \to -\infty,$$

and by the convergence in $L^2(T_0)$,

$$\int_{T_0} (F(x, \tau^1_{-i} u) - F(x, v))\,dx \to 0, \quad i \to -\infty.$$

These observations and (2.9) show that

$$\underline{\lim}_{i \to -\infty} J_{1,i}(u) = \underline{\lim}_{i \to -\infty} \frac{1}{2} \|\nabla(\tau^1_{-i} u - v)\|^2_{L^2(T_0)} \ge 0. \tag{2.30}$$

If $\underline{\lim}_{i \to -\infty} J_{1,i}(u)$ is positive, $J_1(u) = \infty$, contrary to hypothesis. Hence $\underline{\lim}_{i \to -\infty} J_{1,i}(u) = 0$. Providing a similar argument for $i \to \infty$, (2.25) follows with lim replaced by $\underline{\lim}$. Then (2.30) yields (2.26)–(2.27) along a subsequence. Next it will be shown that

$$\text{(i)} \lim_{p \to -\infty} J_{1;p,0}(u) \quad \text{and} \quad \text{(ii)} \lim_{q \to \infty} J_{1;1,q}(u) \tag{2.31}$$

exist. Then (2.25) and (2.28) are valid, and returning to (2.9) again shows that (2.26)–(2.27) hold. A slight variant of an argument from [7] – see also Bosetto and Serra [25] – will be employed.

Their proofs being the same, the existence of (2.31) (i) will be verified. Set

$$\mathcal{P} = \{p \in \mathbb{Z} \mid p < 0 \text{ and } J_{1,p}(u) \leq 0\}.$$

If $\mathcal{P}$ is a finite set, $J_{1;p,0}(u)$ is a monotone nondecreasing sequence with $J_{1;p,0}(u) \leq J_1(u) + 2K_1$. Hence (2.31) (i) follows. If $\mathcal{P}$ is infinite, (2.9) shows that

$$\lim_{i \to -\infty, i \in \mathcal{P}} \|\tau_{-i}^1 u - v\|_{W^{1,2}(T_0)} = 0. \tag{2.32}$$

Suppose $J_{1;p,0}(u)$ does not converge as $p \to -\infty$. Set

$$\ell^- = \varliminf_{p \to -\infty} J_{1;p,0}(u), \quad \ell^+ = \varlimsup_{p \to -\infty} J_{1;p,0}(u),$$

so $-K_1 \leq \ell^- < \ell^+$. Choose

$$0 < \varepsilon < (\ell^+ - \ell^-)/5. \tag{2.33}$$

The following technical lemma is useful at this point.

Lemma 2.34. *For any* $\gamma > 0$, *there is a* $\delta = \delta(\gamma) > 0$ *such that if* $u \in \Gamma_1(v, w)$, $p, q \in \mathbb{Z}$, *with* $p < q$ *and*

$$\|u - v\|_{W^{1,2}(T_j)} \leq \delta \text{ or } \|u - w\|_{W^{1,2}(T_j)} \leq \delta \tag{2.35}$$

for $j = p$ *and* q, *then*

$$J_{1;p+1,q-1}(u) \geq -\gamma. \tag{2.36}$$

Assuming Lemma 2.34 for the moment, choose $\gamma = \varepsilon$ and $\delta = \delta(\varepsilon)$. By (2.30) and (2.32), there is a $p_0 \in \mathcal{P}$ such that

$$\begin{cases} J_{1,p}(u) \geq -\varepsilon & \text{for all } p \leq p_0, \\ \|\tau_{-p}^1 u - v\|_{W^{1,2}(T_0)} \leq \delta, & p \leq p_0, p \in \mathcal{P}. \end{cases} \tag{2.37}$$

Hence by Lemma 2.34,

$$J_{1;p+1,q-1}(u) \geq -\varepsilon, \tag{2.38}$$

whenever $p, q \in \mathcal{P}$ and $p < q \leq p_0$. Choose sequences $(p_k), (q_k) \subset -\mathbb{N}$ such that $q_{k+1} < p_k < q_k < p_0$ and

$$J_{1;p_k,0}(u) \to \ell^-; \quad J_{1;q_k,0}(u) \to \ell^+, \quad k \to \infty. \tag{2.39}$$

Therefore there is a k_0 such that for $k \geq k_0$,

$$J_{1;p_k,0}(u) \leq \ell^- + \varepsilon; \quad J_{1;q_k,0}(u) \geq \ell^+ - \varepsilon. \tag{2.40}$$

Next let $\widehat{q}_k$ be the largest $q \in \mathcal{P}$ such that $q < q_k$ and let $\widehat{p}_k$ be the smallest $p \in \mathcal{P}$ such that $p \geq p_k$. Then

$$J_{1;p_k,\widehat{p}_k-1}(u) \geq 0 \tag{2.41}$$

(where this term is not present if $\widehat{p}_k = p_k$). Thus by (2.40),

$$J_{1;\widehat{p}_k,0}(u) \leq \ell^- + \varepsilon. \tag{2.42}$$

Similarly

$$J_{1;\widehat{q}_k+1,q_k-1}(u) \geq 0, \tag{2.43}$$

so by (2.40) again,

$$J_{1;\widehat{q}_k+1,0}(u) \geq \ell^+ - \varepsilon. \tag{2.44}$$

Consequently, by (2.42), (2.44), and (2.33),

$$J_{1;\widehat{p}_k,\widehat{q}_k}(u) = J_{1;\widehat{p}_k,0}(u) - J_{1;\widehat{q}_k+1,0}(u) \leq \ell^- + \varepsilon - (\ell^+ - \varepsilon) < -3\varepsilon. \tag{2.45}$$

On the other hand, by (2.38),

$$J_{1;\widehat{p}_k+1,\widehat{q}_k-1}(u) \geq -\varepsilon, \tag{2.46}$$

which combined with (2.37) with $p = p_k, q_k$ yields

$$J_{1;\widehat{p}_k,\widehat{q}_k}(u) \geq -3\varepsilon, \tag{2.47}$$

contrary to (2.45). Thus $\ell^+ = \ell^-$ and the proof of Proposition 2.24 is complete modulo the:

Proof of Lemma 2.34. Suppose, e.g., (2.35) holds with the v term. Take χ as in (2.15). Then (2.16) implies the result, provided that

$$|J_{1,p}(\chi)| + |J_{1,q}(\chi)| \leq \gamma. \tag{2.48}$$

But (2.48) follows from (2.35), the form of χ, and the continuity of $J_{1,i}$ (in $\|\cdot\|_{W^{1,2}(T_i)}$) for $i \in \mathbb{Z}$.

Corollary 2.49. *Suppose $u \in \widehat{\Gamma}_1(v,w)$, $J_1(u) < \infty$, and $u \leq \tau_{-1}^1 u$. Then either (i) $u \in \mathcal{M}_0$, or (ii) there are $\varphi, \psi \in \mathcal{M}_0$ with $v \leq \varphi < \psi \leq w$ such that $u \in \Gamma_1(\varphi,\psi)$.*

Proof. Set $u_k = \tau_{-k}^1 u$. Since $J_1(u) < \infty$ and by (2.23) (u_k) is bounded in $W^{1,2}(T_0)$, there is a $\varphi \in W^{1,2}(T_0)$ such that $u_k \to \varphi$ as $k \to -\infty$ along a subsequence, weakly in $W^{1,2}(T_0)$ and strongly in $L^2(T_0)$. Since $\tau_{-1}^1 u_k = u_{k+1} \geq u_k$, the entire sequence converges to φ in $L^2(T_0)$ and $\tau_{-1}^1\varphi = \varphi$, i.e., $\varphi \in \Gamma_0$. If $\varphi \notin \mathcal{M}_0$, $J_0(\varphi) > c_0 + \varepsilon$ for some $\varepsilon > 0$. Since J_0 is weakly lower semicontinuous,

$$c_0 + \varepsilon < J_0(\varphi) \le \varliminf_{k\to\infty} J_0(u_k).$$

But then $J_1(u) = \infty$, a contradiction. Thus φ and similarly ψ, the weak limit of u_k as $k \to \infty$, belong to $\mathcal{M}_0$. If $\varphi = \psi$, $u \le \tau_{-1}^1 u$ implies $u = \varphi$ and (i) holds. Otherwise $\varphi < \psi$ and (ii) is valid.

Having established some convergence results for J_1, next a compactness property of minimizing sequences will be studied. It represents, in the current setting, the analogue of the Palais–Smale condition in other contexts involving critical point theory and is modeled on similar results in [7].

Proposition 2.50. *Let $\mathcal{Y} \subset \widehat{\Gamma}_1(v,w)$. Suppose $\mathcal{Y}$ possesses the following property:*

(Y_1^1) Let $u \in \mathcal{Y}$, $p \in \mathbb{N}$, and let U be a sequential weak limit (in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{T}^{n-1})$) of $(u_k) \subset \mathcal{Y}$. Define $\chi_p \equiv \chi_p(u,U)$ by

$$\chi_p = \begin{cases} u, & x_1 \le -p, \\ U, & -p+1 \le x_1 \le p, \\ u, & p+1 \le x_1, \end{cases}$$

and extend χ_p to the intermediate intervals as in (2.15). Then $\chi_p(u,U) \in \mathcal{Y}$ for all large p (independently of u).

Define

$$c(\mathcal{Y}) = \inf_{u\in\mathcal{Y}} J_1(u). \tag{2.51}$$

If $c(\mathcal{Y}) < \infty$ and (u_k) is a minimizing sequence for (2.51), then there is a $U \in \widehat{\Gamma}_1$ such that along a subsequence, $u_k \to U$ in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{T}^{n-1})$.

Proof. Since (u_k) is a minimizing sequence for (2.51), there is an $M > 0$ such that

$$J_1(u_k) \le M. \tag{2.52}$$

By Lemma 2.22, (u_k) is bounded in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{T}^{n-1})$. Therefore passing to a subsequence, it can be assumed that there is a $U \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{T}^{n-1})$ such that $u_k \to U$ weakly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{T}^{n-1})$, strongly in $L^2_{\mathrm{loc}}(\mathbb{R}\times\mathbb{T}^{n-1})$, and pointwise a.e. Thus $U \in \widehat{\Gamma}_1$. It remains to show that convergence is in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{T}^{n-1})$. For $i \in \mathbb{Z}$, set

$$\delta_i = \varliminf_{s\to\infty} J_{1,i}(u_s) - J_{1,i}(U). \tag{2.53}$$

Since $J_{1,i}$ is weakly lower semicontinuous, $\delta_i \geq 0$. By Proposition 2.8, Lemma 2.22, and (2.53), for any $p \in \mathbb{N}$,

$$-K_1 \leq J_{1;-p,p}(U) = \sum_{-p}^{p}\left(\varliminf_{s\to\infty} J_{1,i}(u_s) - \delta_i\right) \leq \varliminf_{s\to\infty} J_{1;-p,p}(u_s) - \sum_{-p}^{p}\delta_i$$
$$\leq \varliminf_{s\to\infty} J_1(u_s) + 2K_1 - \sum_{-p}^{p}\delta_i. \tag{2.54}$$

Therefore by (2.52) and (2.54),

$$\sum_{i\in\mathbb{Z}}\delta_i \leq M + 3K_1. \tag{2.55}$$

Consequently $\delta_i \to 0$ as $|i| \to \infty$. Next observe that by (2.53), (2.9), and the convergence already established for u_k,

$$\delta_i = \frac{1}{2}\varliminf_{s\to\infty}\left(\|\nabla(u_s - v)\|^2_{L^2(T_i)} - \|\nabla(U - v)\|^2_{L^2(T_i)}\right). \tag{2.56}$$

Since

$$\|\nabla(u_s - U)\|^2_{L^2(T_i)} = \|\nabla(u_s - v)\|^2_{L^2(T_i)} + \|\nabla(U - v)\|^2_{L^2(T_i)}$$
$$- 2\int_{T_i}\nabla(u_s - v)\cdot\nabla(U - v)dx,$$

$$\varliminf_{s\to\infty}\|\nabla(u_s - U)\|^2_{L^2(T_i)} = \varliminf_{s\to\infty}\|\nabla(u_s - v)\|^2_{L^2(T_i)} - \|\nabla(U - v)\|^2_{L^2(T_i)}. \tag{2.57}$$

Thus combining (2.56)–(2.57) yields

$$2\delta_i = \varliminf_{s\to\infty}\|\nabla(u_s - U)\|^2_{L^2(T_i)}. \tag{2.58}$$

By (Y_1^1), $\chi_{k,p} \equiv \chi_p(u_k, U) \in \mathcal{Y}$ for large p. Therefore

$$c(\mathcal{Y}) \leq J_1(\chi_{k,p}) = J_{1;-\infty,-p}(u_k) + J_{1;-p+1,p-1}(U) + J_{1;p,\infty}(u_k)$$
$$+ J_{1,-p}(\chi_{k,p}) - J_{1,-p}(u_k) + J_{1,p}(\chi_{k,p}) - J_{1,p}(u_k). \tag{2.59}$$

Passing to a subsequence of (u_k) for which (2.58) holds as a limit, it follows that there is an $\alpha_p \to 0$ as $p \to \infty$ such that

$$|J_{1,-p}(\chi_{k,p}) - J_{1,-p}(u_k)| + |J_{1,p}(\chi_{k,p}) - J_{1,p}(u_k)| \leq \alpha_p \tag{2.60}$$

for $k \geq k_0(p)$. Thus by (2.59)–(2.60),

$$\begin{aligned} c(\mathcal{Y}) &\leq J_1(u_k) + J_{1;-p+1,p-1}(U) - J_{1;-p+1,p-1}(u_k) + \alpha_p \\ &\leq J_1(u_k) + \varliminf_{s\to\infty} J_{1;-p+1,p-1}(u_s) - J_{1;-p+1,p-1}(u_k) - \sum_{-p+1}^{p-1} \delta_i + \alpha_p. \quad (2.61) \end{aligned}$$

Letting $k \to \infty$ gives

$$\sum_{-p+1}^{p-1} \delta_i \leq \alpha_p, \tag{2.62}$$

and then letting $p \to \infty$ shows that $\delta_i = 0$ for all $i \in \mathbb{Z}$, completing the proof of Proposition 2.50.

Remark 2.63. For the results of Chapters 3–5, the choice of $\mathcal{Y}$ is such that a milder version of (Y_1^1) suffices: There is an $R > 0$ such that whenever $u \in \mathcal{Y}$ and $\chi \in \widehat{\Gamma}_1$ with $\chi(x) = u(x)$ for $|x_1| \geq R$, then $\chi \in \mathcal{Y}$. However, for the results of the later sections involving multitransition solutions, the sets $\mathcal{Y}$ used there involve additional integral constraints. These constraints are also satisfied by the weak $W^{1,2}_{\mathrm{loc}}$ limits of sequences of $\mathcal{Y}$, and the full strength of (Y_1^1) is needed for these settings.

In applications of Proposition 2.50 in later sections, the members of $\mathcal{Y}$ will satisfy some asymptotic conditions as in the definition of Γ_1. The convergence of u_k to U is merely in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{T}^{n-1})$ so a priori U need not possess this asymptotic behavior. Consequently, U may not belong to $\mathcal{Y}$. Nevertheless, if an additional condition is satisfied by the minimizing sequence, U will satisfy (PDE), as the next result shows.

Proposition 2.64. *Under the hypothesis of Proposition 2.50, suppose*

(Y_2^1) there is a minimizing sequence (u_k) for (2.51) such that for some $r \in (0, \frac{1}{2})$, some $z \in \mathbb{R}^n$, all smooth φ with support in $B_r(z) = \{x \in \mathbb{R}^n \mid |x-z| < r\}$ and associated $t_0(\varphi) > 0$,

$$c(\mathcal{Y}) \leq J_1(u_k + t\varphi) + \delta_k \tag{2.65}$$

for all $|t| \leq t_0(\varphi)$, where $\delta_k = \delta_k(\varphi) \to 0$ as $k \to \infty$.

Then the weak limit U of u_k satisfies (PDE) in $B_r(z)$.

Proof. Suppose (u_k) is the minimizing sequence for (2.51) satisfying (2.65). Define ε_k via

$$J_1(u_k) = c(\mathcal{Y}) + \varepsilon_k, \tag{2.66}$$

so $\varepsilon_k \to 0$ as $k \to \infty$. By (2.65),

$$c(\mathcal{Y}) \leq J_1(u_k) = c(\mathcal{Y}) + \varepsilon_k \leq J_1(u_k + t\varphi) + \delta_k + \varepsilon_k,$$

or

$$J_1(u_k) \le J_1(u_k + t\varphi) + \delta_k + \varepsilon_k. \tag{2.67}$$

Now $B_r(z) \subset [p, q+1] \times \mathbb{T}^{n-1}$ for some $p, q \in \mathbb{Z}$, $p \le q$. Then by (2.67),

$$J_{1;p,q}(u_k) \le J_{1:p,q}(u_k + t\varphi) + \delta_k + \varepsilon_k. \tag{2.68}$$

Letting $k \to \infty$ and using the $W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{T}^{n-1})$ convergence of u_k to U, (2.68) shows that

$$J_{1;p,q}(U) \le J_{1;p,q}(U + t\varphi),$$

or

$$\int_{B_r(z)} L(U)dx \le \int_{B_r(z)} L(U + t\varphi)dx. \tag{2.69}$$

for all smooth φ with support in $B_r(z)$ and $|t| \le t_0(\varphi)$. Hence standard elliptic regularity arguments imply that U is a solution of (PDE) in $B_r(z)$.

The final result in this section provides a useful tool for comparison arguments that will be used repeatedly later. For $v \in \mathcal{M}_0$, set

$$\Gamma_1(v) = \{u \in \widehat{\Gamma}_1(v-1, v+1) \mid \|u - v\|_{L^2(T_i)} \to 0,\ |i| \to \infty\}.$$

Remark 2.70. It is readily verified that the conclusions of Proposition 2.24 hold for $\Gamma_1(v)$, (2.27) being deleted and (2.26) valid for $|i| \to \infty$.

Define

$$c_1(v) = \inf_{u \in \Gamma_1(v)} J_1(u) \tag{2.71}$$

and set

$$\mathcal{M}_1(v) = \{u \in \Gamma_1(v) \mid J_1(u) = c_1(v)\}.$$

Theorem 2.72. *If F satisfies (F_1)–(F_2), then $c_1(v) = 0$ and $\mathcal{M}_1(v) = \{v\}$.*

Proof. Since $v \in \Gamma_1(v)$ and $J_1(v) = 0$,

$$c_1(v) \le 0. \tag{2.73}$$

To obtain the reverse inequality, it suffices to show that

$$J_1(u) \ge 0 \tag{2.74}$$

for all $u \in \Gamma_1(v)$. Thus suppose $u \in \Gamma_1(v)$ and $J_1(u) < \infty$. In the definition of χ_p in (Y_1^1) of Proposition 2.50, replace u by v, U by u, and denote the resulting function by χ_p. Thus $\chi_p \in \Gamma_1(v)$. Set $\varphi_p = \chi_p|_{[-p-1,p+1]\times\mathbb{T}^{n-1}}$ and extended as a $(2p+2)$-periodic function of x_1. Then $\varphi_p \in \Gamma_0(\ell)$ with $\ell = (2p+2, 0, \ldots, 0)$, so by Proposition 2.2,

$$0 \le J_{1;-p-1,p}(\varphi_p) = J_{1;-p,p}(\varphi_p) = J_{1;-p,p}(\chi_p) = J_1(\chi_p). \tag{2.75}$$

Now

$$\begin{aligned} J_1(\chi_p) &= J_1(u) + J_{1,-p}(\chi_p) - J_{1,-p}(u) \\ &\quad + J_{1,p}(\chi_p) - J_{1,p}(u) - J_{1;-\infty,-p-1}(u) - J_{1;p+1,\infty}(u) \\ &\equiv J_1(u) - R_p(u), \end{aligned}$$

so by (2.75),

$$R_p(u) \leq J_1(u). \tag{2.76}$$

Now to prove (2.74), it will be shown that $R_p(u) \to 0$ as $p \to \infty$. By Remark 2.70 and Proposition 2.24, the tails $J_{1;-\infty,-p-1}(u)$, $J_{1;p+1,\infty}(u)$ approach 0 as $p \to \infty$ and likewise the differences

$$J_{1;-p}(\chi_p) - J_{1,-p}(u), \quad J_{1,p}(\chi_p) - J_{1,p}(u)$$

go to 0 as $p \to \infty$, since $\tau^1_{\pm p}\chi_p$, $\tau^1_{\pm p}u \to v$ in $W^{1,2}(T_0)$ via (2.26).

Remark 2.77. The above argument holds equally well if $v \pm 1$ is replaced by $v \pm j$ for any $j \in \mathbb{N}$.

It remains to prove that $\mathcal{M}_1(v) = \{v\}$. Let $u \in \mathcal{M}_1(v)$. Then $v-1 \leq u \leq v+1$, so for any $z \in \mathbb{R}^n$, $r \in (0, \frac{1}{2})$, φ smooth with support in $B_r(z)$, and $|t|$ small (depending on φ), $v-2 \leq u+t\varphi \leq v+2$. Hence with the aid of Remark 2.77, and $u_k = u$, note that (Y^1_2) of Proposition 2.64 (with $\delta_k = 0$) is satisfied. Consequently, u satisfies (PDE) for all $z \in \mathbb{R}^n$. By (F_2), $u \in \mathcal{M}_1(v)$ implies $\tau^1_{-1}u \in \mathcal{M}_1(v)$. If $\tau^1_{-1}u = u$, u is 1-periodic in x_1, and $\|u-v\|_{L^2(T_i)} \to 0$ as $|i| \to \infty$ then implies $u \equiv v$, completing the proof. Thus suppose $u \neq \tau^1_{-1}u$. An argument like that of Proposition 2.2 (and essentially due to Moser [1]) then leads to a contradiction. We claim that

$$\text{(i) } u < \tau^1_{-}u \quad \text{or} \quad \text{(ii) } u > \tau^1_{-1}u. \tag{2.78}$$

Otherwise, set $\varphi = \max(u, \tau^1_{-1}u)$ and $\psi = \min(u, \tau^1_{-1}u)$. Then $\varphi \geq \psi$ and there are points ξ and η such that $\varphi(\xi) = \psi(\xi)$ and $\varphi(\eta) > \psi(\eta)$. Note that for any $i \in \mathbb{Z}$,

$$\int_{T_i} (L(\varphi) + L(\psi))dx = \int_{T_i} (L(u) + L(\tau^1_{-1}u))dx,$$

or

$$J_{1,i}(\varphi) + J_{1,i}(\psi) = J_{1,i}(u) + J_{1,i}(\tau^1_{-1}u). \tag{2.79}$$

Therefore summing over i leads to

$$J_1(\varphi) + J_1(\psi) = J_1(u) + J_1(\tau^1_{-1}u) = 0. \tag{2.80}$$

Since $\varphi, \psi \in \Gamma_1(v)$, $J_1(\varphi)$, $J_1(\psi) \geq 0$. Hence by (2.80), $\varphi, \psi \in \mathcal{M}_1(v)$ and thus they satisfy (PDE). Consequently their difference $f = \varphi - \psi$ is nonnegative and satisfies (2.5). Hence a contradiction as in the proof of Proposition 2.2 obtains, yielding (2.78). The remaining argument is the same for (i) or (ii) in (2.78), so suppose (i) holds. Then for all $j \in \mathbb{N}$,

$$\tau_j^1 u < u < \tau_{-j}^1 u. \tag{2.81}$$

Letting $j \to \infty$ gives

$$v \le u \le v, \tag{2.82}$$

and the proof of Theorem 2.72 is complete.

Remark 2.83. Suppose $(*)_0$ holds. Set

$$\widetilde{\Gamma}_1(v_0) = \{u \in \Gamma_1(v_0) \mid v_0 \le u \le w_0\}$$

and

$$\widetilde{c}_1(v_0) = \inf_{u \in \widetilde{\Gamma}_1(v_0)} J_1(u).$$

Then since $\widetilde{\Gamma}_1(v_0) \subset \Gamma_1(v_0)$,

$$0 = c_1(v_0) \le \widetilde{c}_1(v_0) \le J_1(v_0) = 0, \tag{2.84}$$

so $\widetilde{c}_1(v_0) = 0$ and likewise

$$\widetilde{\mathcal{M}}_1(v_0) = \{u \in \widetilde{\Gamma}_1(v_0) \mid J_1(u) = \widetilde{c}_1(v_0)\} = \{v_0\}$$

via Theorem 2.72.

Remark 2.85. Suppose condition (F_3) holds, i.e., F is even in $x_1, \dots, x_n$. Then as was shown in [9],

$$c_0 = \inf_{u \in W^{1,2}([0,1]^n)} J_0(u)$$

and any $u \in \mathcal{M}_0$ is even in $x_1, \dots, x_n$. Therefore if $u \in \widehat{\Gamma}_1(v_0, w_0)$,

$$J_{1,i}(u) = \int_{T_i} L(u)dx - c_0 \ge 0$$

for all $i \in \mathbb{Z}$ and $J_1(u) \ge 0$ on this set of functions. This fact allows us to obtain several of the results of this section and in the sequel much more simply. See, e.g., [13, 14] for a treatment of (PDE) under this additional hypothesis.

Chapter 3
The Simplest Heteroclinics

Using the preliminary results of Chapter 2, the existence of heteroclinic solutions of (PDE) will be established in this section. To formulate the main result, set

$$c_1 = c_1(v_0, w_0) \equiv \inf_{u \in \Gamma_1(v_0, w_0)} J_1(u). \tag{3.1}$$

Theorem 3.2. *If F satisfies (F_1)–(F_2) and $(*)_0$ holds,*

1^o There is a $U_1 \in \Gamma_1$ such that $J_1(U_1) = c_1$, i.e.,

$$\mathcal{M}_1 = \mathcal{M}_1(v_0, w_0) \equiv \{u \in \Gamma_1(v_0, w_0) \mid J_1(u) = c_1\} \neq \emptyset;$$

2^o Any $U \in \mathcal{M}_1$ satisfies

(a) U is a solution of (PDE);
(b) $\|U - v_0\|_{C^2(T_i)} \to 0$, $i \to -\infty$,
$\|U - w_0\|_{C^2(T_i)} \to 0$, $i \to \infty$,
i.e., U is heteroclinic in x_1 from v_0 to w_0,
(c) $v_0 < U < \tau_{-1}^1 U < w_0$, i.e., U is strictly 1*-monotone in x_1,*

3^o $\mathcal{M}_1$ is an ordered set.

Proof. Let $(u_k) \subset \Gamma_1$ be a minimizing sequence for (3.1). Then there is an $M > 0$ such that for all $k \in \mathbb{N}$,

$$J_1(u_k) \leq M. \tag{3.3}$$

Since $J_1(u) = J_1(\tau_{-1}^1 u)$ for $u \in \Gamma_1$ via (F_2), unless a normalization is imposed on (u_k), it may converge weakly to, e.g., v_0, yielding no useful information. Thus normalize u_k via

$$\int_{T_i} u_k \, dx \leq \frac{1}{2} \int_{T_0} (v_0 + w_0) dx \leq \int_{T_0} u_k \, dx \tag{3.4}$$

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for all $i \in \mathbb{Z}$, $i < 0$, and for all $k \in \mathbb{N}$. This is possible by the definition of Γ_1. Noting that $\mathcal{Y} = \Gamma_1(v_0, w_0)$ satisfies (Y_1^1) of Proposition 2.50, by that result there is a $U_1 \in \widehat{\Gamma}_1(v_0, w_0)$ such that $u_k \to U_1$ in $W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ along a subsequence that can be taken to be the entire sequence. By (3.4), for $0 > i \in \mathbb{Z}$,

$$\int_{T_i} U_1 \, dx \le \frac{1}{2} \int_{T_0} (v_0 + w_0) dx \le \int_{T_0} U_1 \, dx, \tag{3.5}$$

so $v_0 \not\equiv U_1 \not\equiv w_0$. By Proposition 2.8, (2.23), and the weak lower semicontinuity of $J_{1;p,q}$,

$$-K_1 \le J_{1;p,q}(U_1) \le M + 2K_1 \tag{3.6}$$

for any $p \le q$. Hence

$$-K_1 \le J_1(U_1) \le M + 2K_1. \tag{3.7}$$

To complete the proof, it will be shown that (A) U_1 is a solution of (PDE), as is any $U \in \mathcal{M}_1$; (B) U_1 and any $U \in \mathcal{M}_1$ satisfy 2^o(b) and 2^o(c); (C) $J_1(U_1) = c_1$, so 1^o holds, and lastly (D) 3^o is valid.

Proof of (A). For the first statement it suffices to verify (Y_2^1) of Proposition 2.64 for (u_k). Since $v_0 \le u_k \le w_0$, for $t_0 = t_0(\varphi)$ sufficiently small,

$$w_0 - 2 \le v_0 - 1 \le u_k + t\varphi \le w_0 + 2.$$

Set $f_k = \max(u_k + t\varphi, w_0)$ and $g_k = \min(u_k + t\varphi, w_0)$. By Remark 2.77, it can be assumed that $f_k \in \Gamma_1(w_0)$. Hence by Theorem 2.72,

$$J_1(f_k) \ge 0. \tag{3.8}$$

Since $g_k \in \widehat{\Gamma}_1(v_0 - 1, w_0)$, by (3.8),

$$J_1(g_k) \le J_1(f_k) + J_1(g_k), \tag{3.9}$$

and as in (2.79)–(2.80),

$$J_1(f_k) + J_1(g_k) = J_1(u_k + t\varphi). \tag{3.10}$$

Set $\chi_k = \max(g_k, v_0)$ and $\psi_k = \min(g_k, v_0)$. Then $\chi_k \in \Gamma_1$ and $\psi_k \in \Gamma(v_0)$, so as in (3.8)–(3.10),

$$J_1(\chi_k) \le J_1(\chi_k) + J_1(\psi_k) = J_1(g_k). \tag{3.11}$$

Combining (3.9)–(3.11) gives

$$c_1 \le J_1(u_k) \equiv c_1 + \delta_k \le J_1(\chi_k) + \delta_k \le J_1(u_k + t\varphi) + \delta_k, \tag{3.12}$$

where $\delta_k \to 0$ as $k \to \infty$. Thus (Y_2^1) holds and U_1 is a solution of (PDE). Next observe that if $U \in \mathcal{M}_1$, the sequence (φ_k), where $\varphi_k = U$ for all $k \in \mathbb{N}$, is a minimizing sequence for (3.1). Hence by what was just shown, U is a solution of (PDE).

Proof of (B). Suppose that

$$U_1 \le \tau_{-1}^1 U_1. \tag{3.13}$$

Then since $U_1 \in \widehat{\Gamma}_1(v_0, w_0) \setminus \{v_0, w_0\}$ by (3.7), (3.13), and Corollary 2.49, $U_1 \in \Gamma_1(v_0, w_0)$. Likewise any $U \in \mathcal{M}_1$ belongs to $\Gamma_1(v_0, w_0)$. Hence by Proposition 2.24, $\|u - v_0\|_{W^{1,2}(T_i)} \to 0$ as $i \to -\infty$ and $\|u - w_0\|_{W^{1,2}(T_i)} \to 0$ as $i \to \infty$ for $u = U_1$ or U. By (A), u is a solution of (PDE), and by the Schauder estimates, for any $\alpha \in (0, 1)$, u is bounded in $C^{2,\alpha}(\mathbb{R} \times \mathbb{T}^{n-1})$. Hence the $W^{1,2}$ asymptotics and standard interpolation inequalities yield 2^o(b).

To verify (3.13), set $\Phi_k = \max(u_k, \tau_{-1}^1 u_k)$ and $\Psi_k = \min(u_k, \tau_{-1}^1 u_k)$. Then $\Phi_k, \Psi_k \in \Gamma_1$ and as in (2.79)–(2.80),

$$J_1(\Phi_k) + J_1(\Psi_k) = J_1(u_k) + J_1(\tau_{-1}^1 u_k) = 2J_1(u_k) \to 2c_1 \tag{3.14}$$

as $k \to \infty$. Therefore Φ_k and Ψ_k are also minimizing sequences for (3.1). Using Propositions 2.50 and 2.64 again together with the fact that $\max(\cdot, \cdot)$ and $\min(\cdot, \cdot)$ are continuous on $W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ shows that $\Phi_k \to \Phi = \max(U_1, \tau_{-1}^1 U_1)$ and $\Psi_k \to \Psi = \min(U_1, \tau_{-1}^1 U_1)$ in $W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ as $k \to \infty$ with Φ, Ψ solutions of (PDE). Since $\Phi \ge \Psi$, the maximum principle argument following (2.5) implies either (i) $\Phi \equiv \Psi$ or (ii) $\Phi > \Psi$ on $\mathbb{R} \times \mathbb{T}^{n-1}$. If (i) occurs, U_1 is 1-periodic in x_1, so $U_1 \in \Gamma_0 \cap \widehat{\Gamma}_1$. Moreover, as noted earlier, $v_0 \not\equiv U_1 \not\equiv w_0$. Therefore $J_0(U_1) > c_0$, so $J_1(U_1) = \infty$, contrary to (3.7). Thus (ii) occurs, so either (a) $U_1 > \tau_{-1}^1 U_1$ or (b) $U_1 < \tau_{-1}^1 U_1$. But (a) is not compatible with (3.5) (for $i = -1$). Therefore (b) holds and (3.13) is valid. Note also that if $U \in \mathcal{M}_1$ and Φ, Ψ are as above with U replacing U_1, (3.14) with $J_1(U) = c_1$ shows that Φ and Ψ are solutions of (PDE). Again as above this leads to (a) or (b), and since $U \in \Gamma_1$, its asymptotic behavior excludes (a). Thus any $U \in \mathcal{M}_1$ also satisfies (3.13). The remaining inequalities in 2^o(c) for U_1 or $U \in \mathcal{M}_1$ follow from the maximum principle as in (2.5).

Proof of (C). Since $U_1 \in \Gamma_1$,

$$J_1(U_1) \ge c_1. \tag{3.15}$$

An approximation argument will be used to obtain the reverse inequality. By (3.7), $J_1(U_1) < \infty$ so Proposition 2.24 implies

$$\begin{cases} \|U_1 - v_0\|_{W^{1,2}(T_i)} \to 0, & i \to -\infty, \\ \|U_1 - w_0\|_{W^{1,2}(T_i)} \to 0, & i \to \infty. \end{cases} \tag{3.16}$$

Set $\widehat{T}_i = \bigcup_{j=i-1}^{i+1} T_j$. Let $\varepsilon > 0$.

By (3.16), $p_0 = p_0(\varepsilon)$ can be chosen such that if $p \ge p_0$,

$$\|U_1 - v_0\|_{W^{1,2}(\widehat{T}_{-p})} \le \varepsilon/2. \tag{3.17}$$

By Proposition 2.50, there is a $k_0 = k_0(p)$ such that for $k \geq k_0$,

$$\|u_k - U_1\|_{W^{1,2}(\widehat{T}_{-p})} \leq \varepsilon/2. \tag{3.18}$$

Hence for such k and p,

$$\|u_k - v_0\|_{W^{1,2}(\widehat{T}_{-p})} \leq \varepsilon. \tag{3.19}$$

Similarly for $k \geq k_0(p)$, it can be assumed that

$$\|u_k - w_0\|_{W^{1,2}(\widehat{T}_p)} \leq \varepsilon. \tag{3.20}$$

Since $u_k \in \Gamma_1$, there is a $q_0 = q_0(k)$ such that for $q \geq q_0$,

$$\|u_k - v_0\|_{W^{1,2}(\widehat{T}_{-q})}, \quad \|u_k - w_0\|_{W^{1,2}(\widehat{T}_q)} \leq \varepsilon. \tag{3.21}$$

Define

$$f_k = \begin{cases} u_k, & x_1 \leq p-1, \\ w_0, & p \leq x_1 \leq p+1, \\ u_k, & p+2 \leq x_1 \leq q-1, \\ w_0, & q \leq x_1 \leq q+1, \\ u_k, & q+2 \leq x_1. \end{cases} \tag{3.22}$$

Extend f_k to the intermediate intervals as in (2.15). Then by (3.20)–(3.21), there is a $\kappa(\varepsilon)$ such that

$$|J_{1;p,q}(u_k) - J_{1;p,q}(f_k)| \leq \kappa(\varepsilon) \tag{3.23}$$

and $\kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$. The function $f_k\big|_{(p,q+1)\times\mathbb{T}^{n-1}}$ extends naturally to a $(q+1-p)$-periodic function of x_1, so by Proposition 2.2,

$$J_{1;p,q}(f_k) \geq 0. \tag{3.24}$$

Since

$$J_{1;1,\infty}(u_k) = J_{1;1,p-1}(u_k) + J_{1;p,q}(u_k) + J_{1;q+1,\infty}(u_k), \tag{3.25}$$

by (3.23)–(3.25),

$$J_{1;1,\infty}(u_k) \geq J_{1;1,p-1}(u_k) - \kappa(\varepsilon) + J_{1;q+1,\infty}(u_k). \tag{3.26}$$

Letting $q \to \infty$ in (3.26) and combining it with the analogous estimate for $J_{1;-\infty,0}(u_k)$ yields

$$J_1(u_k) \geq J_{1;-p+1,p-1}(u_k) - 2\kappa(\varepsilon). \tag{3.27}$$

Thus letting $k \to \infty$ and using Proposition 2.50 again shows that

$$c_1 \geq J_{1;-p+1,p-1}(U_1) - 2\kappa(\varepsilon). \tag{3.28}$$

Lastly, letting $p \to \infty$ and then $\varepsilon \to 0$ gives

$$c_1 \geq J_1(U_1). \tag{3.29}$$

This with (3.15) completes the proof of 1^o of Theorem 3.2.

Proof of (D). Let $V, W \in \mathcal{M}_1$ and set $\Phi = \max(V, W)$ and $\Psi = \min(V, W)$. Then

$$J_1(\Phi) + J_1(\Psi) = J_1(V) + J_1(W) = 2c_1, \tag{3.30}$$

so by the argument of the end of (B), $\Phi, \Psi \in \mathcal{M}_1$ with $\Phi \geq \Psi$ and either $\Phi \equiv \Psi$, in which case $V \equiv W$, or $\Phi > \Psi$, and then $V > W$ or $W > V$. The proof of Theorem 3.2 is complete.

Remark 3.31. Reversing the roles of v_0 and w_0, there is also a solution of (PDE) heteroclinic in x_1 from w_0 to v_0 and periodic in $x_2, \ldots, x_n$. Using the natural notation, it lies in

$$\mathcal{M}_1(w_0, v_0) \equiv \{u \in \Gamma_1(w_0, v_0) | J_1(u) = c_1(w_0, v_0)\}.$$

Next we will give another characterization of c_1. For that purpose, set

$$\mathcal{S}_1 = \{u \in \widehat{\Gamma}_1 \mid u \leq \tau_{-1}^1 u \text{ and } v_0 \not\equiv u \not\equiv w_0\}.$$

Corollary 3.32.

$$c_1 = \inf_{u \in \mathcal{S}_1} J_1(u).$$

Proof. By Corollary 2.49, $\{u \in \mathcal{S}_1 \mid J_1(u) < \infty\} \subset \Gamma_1$. Therefore

$$s \equiv \inf_{u \in \mathcal{S}_1} J_1(u) \geq c_1.$$

But by Theorem 3.2, $U \in \mathcal{M}_1$ implies $U \in \mathcal{S}_1$, so $J_1(U) = c_1 \geq s$. Hence $s = c_1$.

Remark 3.33. An examination of the proof of Theorem 3.2 shows that assertions 2^o–3^o do not require $(*)_0$ directly but merely that $\mathcal{M}_1 \neq \emptyset$. Next we will show that $(*)_0$ is both necessary and sufficient in order that $\mathcal{M}_1 \neq \emptyset$.

Theorem 3.34. *Suppose F satisfies (F_1)–(F_2), and $v, w \in \mathcal{M}_0$ with $v \neq w$. Then $\mathcal{M}_1(v, w) \neq \emptyset$ iff v and w are adjacent members of $\mathcal{M}_0$.*

Proof. The sufficiency follows from Theorem 3.2. Thus assume $\mathcal{M}_1(v, w) \neq \emptyset$ with, e.g., $v < w$. If v and w are not adjacent members of $\mathcal{M}_0$, there is a $u \in \mathcal{M}_0$ with $v < u < w$. Let $U \in \mathcal{M}_1(v, w)$. Then $f = \min(u, U) \in \Gamma_1(v, u)$ and $g = \max(u, U) \in \Gamma_1(u, w)$, so as in (2.79)–(2.80),

$$c_1(v, u) + c_1(u, w) \leq J_1(f) + J_1(g) = J_1(U) = c_1(v, w). \tag{3.35}$$

We claim that there is strict equality in (3.35). Otherwise,

$$\varepsilon = c_1(v,w) - c_1(v,u) - c_1(u,v) > 0.$$

With σ free for the moment, choose $\varphi \in \Gamma_1(v,u)$ and $\psi \in \Gamma_1(u,w)$ such that

$$\begin{cases} \|\varphi - u\|_{W^{1,2}(T_i)} \le \sigma, & i \ge 0, \\ \|\psi - u\|_{W^{1,2}(T_i)} \le \sigma, & i \le 1, \end{cases} \tag{3.36}$$

and

$$\begin{cases} J_{1;-\infty,-1}(\varphi) < c_1(v,u) + \varepsilon/3, \\ J_{1;1,\infty}(\psi) < c_1(u,w) + \varepsilon/3. \end{cases} \tag{3.37}$$

This is possible since $h \in \Gamma_1(\alpha,\beta)$ implies $\tau_{-j}^1 h \in \Gamma_1(\alpha,\beta)$ for all $j \in \mathbb{Z}$. Set

$$\chi = \begin{cases} \varphi, & x_1 \le 0, \\ \psi, & x_1 \ge 1, \end{cases} \tag{3.38}$$

with the usual interpolation in between. Choose σ so small that

$$J_{1,0}(\chi) < \varepsilon/3. \tag{3.39}$$

Then by (3.37)–(3.39) and the choice of ε,

$$J_1(\chi) < c_1(v,w). \tag{3.40}$$

But $\chi \in \Gamma_1(v,w)$, so (3.40) is impossible. Hence there is equality in (3.35), so $f \in \mathcal{M}_1(v,u)$ and $g \in \mathcal{M}_1(u,w)$. Therefore by Remark 3.33, f and g are solutions of (PDE) and $v < f < u$. But $f = u$ for large x_1, a contradiction. Consequently $\mathcal{M}_1(v,w) = \emptyset$.

Remark 3.41. Theorem 3.34 does not exclude the possibility of there being a solution of (PDE) heteroclinic in x_1 from v to w with v and w nonadjacent members of $\mathcal{M}_0$. Indeed, such heteroclinics will be constructed in Chapter 9. Theorem 3.34 simply prohibits such solutions from being minimizers of J_1 in $\Gamma_1(v,w)$.

The next result shows that the gap condition $(*)_0$ depends continuously on F. First some notation is needed to deal with multiple functions and functionals associated with (F_1)–(F_2). Suppose H satisfies (F_1)–(F_2). For $u \in \Gamma_0$, set

$$J_0^H(u) = \int_{\mathbb{T}^n} \left(\frac{1}{2}|\nabla u|^2 + H(x,u) \right) dx;$$

$$c_0(H) = \inf_{u \in \Gamma_0} J_0^H(u);$$

and

$$\mathcal{M}_0(H) = \{u \in \Gamma_0 \mid J_0^H(u) = c_0(H)\}.$$

When $(*)_0$ holds for H, an associated gap pair will be denoted by $v_0(H)$, $w_0(H)$.

Proposition 3.42. *Let F satisfy (F_1)–(F_2). If $(*)_0$ holds for F, there is an ε such that if*

$$\|F - \overline{F}\|_{L^\infty(\mathbb{T}^{n+1})} + \|F_u - \overline{F}_u\|_{L^\infty(\mathbb{T}^{n+1})} \leq \varepsilon, \tag{3.43}$$

then $(*)_0$ holds for $\overline{F}$. Moreover, suppose v_0, w_0 is a gap pair for F and

$$\alpha_0 = \int_{T_0} v_0\,dx; \quad \beta_0 = \int_{T_0} w_0\,dx.$$

Then for any $\delta \in (0, \frac{\beta_0 - \alpha_0}{2})$, there is an $\varepsilon_1 = \varepsilon_1(F, \delta)$ such that (3.43) with ε_1 implies

$$\int_{T_0} v\,dx \notin (\alpha_0 + \delta, \beta_0 - \delta) \tag{3.44}$$

for all $v \in \mathcal{M}_0(\overline{F})$.

Proof. If suffices to prove the second assertion. If it is false, for some such δ there is a sequence (F_k) satisfying (F_1)–(F_2),

$$\|F - F_k\|_{L^\infty(\mathbb{T}^{n+1})} + \|F_u - F_{ku}\|_{L^\infty(\mathbb{T}^{n+1})} \leq \frac{1}{k}, \tag{3.45}$$

and an associated $u_k \in \mathcal{M}_0(F_k)$ with

$$\int_{T_0} u_k\,dx \in (\alpha_0 + \delta, \beta_0 - \delta). \tag{3.46}$$

By (3.45), if $w \in \mathcal{M}_0(F)$,

$$\begin{aligned} c_0(F_k) &= \int_{T_0} \left(\frac{1}{2}|\nabla u_k|^2 + F_k(x, u_k)\right) dx \leq \int_{T_0} \left(\frac{1}{2}|\nabla w|^2 + F_k(x, w)\right) dx \\ &= c_0(F) + \int_{T_0} (F_k(x, w) - F(x, w))dx \leq c_0(F) + 1. \end{aligned} \tag{3.47}$$

Therefore

$$\|u_k\|_{W^{1,2}(T_0)} \leq M_1, \tag{3.48}$$

where the constant M_1 is independent of k. By the Poincaré inequality, for any $p > 1$,

$$\|u_k - \int_{T_0} u_k\|_{L^p(T_0)} \le M_2 \|\nabla u_k\|_{L^p(T_0)},$$

and hence by (3.46),

$$\|u_k\|_{L^p(T_0)} \le |\alpha_0| + |\beta_0| + M_2 \|\nabla u_k\|_{L^p(T_0)}. \tag{3.49}$$

Now (PDE) for F_k and the L^p elliptic theory imply

$$\|u_k\|_{W^{2,p}(T_0)} \le M_3(\|u_k\|_{L^p(T_0)} + \|F_{ku}(\cdot, u_k)\|_{L^p(T_0)}). \tag{3.50}$$

By the Gagliardo–Nirenberg inequality [26],

$$\|\nabla u_k\|_{L^p(T_0)} \le M_4 \|u_k\|^a_{w^{2,p}(T_0)} \|\nabla u_k\|^{1-a}_{L^2(T_0)}, \tag{3.51}$$

where $a = n(p-2)/(n(p-2)+2p) \in (0,1)$. Finally, for any $\alpha \in (0,1)$ and p sufficiently large,

$$\|u_k\|_{C^{1,\alpha}(T_0)} \le M_5 \|u_k\|_{W^{2,p}(T_0)}. \tag{3.52}$$

Consequently, (3.48)–(3.52) yield

$$\|u_k\|_{C^{1,\alpha}(T_0)} \le M_6 \tag{3.53}$$

with M_6 independent of k. Hence there is a $u \in C^{1,\alpha}(T_0)$ such that along a subsequence, $u_k \to u$ in $C^1(T_0)$. By (PDE) for F_k,

$$\int_{T_0} (\nabla u \cdot \nabla \varphi + F_u(x,u)\varphi) dx = 0 \tag{3.54}$$

for all $\varphi \in W^{1,2}(T_0)$, i.e., u is a weak solution of (PDE). Standard regularity results therefore imply that $u \in C^{2,\alpha}(T_0)$ and that u is a classical solution of (PDE). Moreover, by (3.45) and (3.47), $J_0(u) = c_0(F)$, so $u \in \mathcal{M}_0(F)$. But by (3.46),

$$\int_{T_0} u \, dx \in (\alpha_0, \beta_0),$$

contrary to $(*)_0$ for F.

Remark 3.55. By (3.44), there is a $v_0(F) \in \mathcal{M}_0(F)$ with largest mean value that is less than $\alpha_0 + \delta$ and a $w_0(F) \in \mathcal{M}_0(F)$ with smallest mean value that is greater than $\beta_0 - \delta$. The proof of Proposition 3.42 shows that the unique gap pair $v_0(\overline{F})$, $w_0(\overline{F})$ for $(*)_0$ for $\overline{F}$ approaches $v_0(F)$, $w_0(F)$ as $\overline{F} \to F$ in $C^1(\mathbb{T}^{n+1})$.

Even if $(*)_0$ fails, by perturbing F slightly, $(*)_0$ can be regained, i.e., $(*)_0$ is a generic condition. More precisely:

Proposition 3.56. *Let F satisfy (F_1)–(F_2). Then there is a G satisfying (F_1)–(F_2) such that if $\varepsilon \neq 0$, $(*)_0$ holds for (PDE) with F replaced by $F + \varepsilon G$.*

Proof. Let $v \in \mathcal{M}_0(F)$ and set

$$G(x,u) = \sin^2 \pi(u - v(x)).$$

Then G satisfies (F_1)–(F_2), and so does $F + \varepsilon G$ for any $\varepsilon \neq 0$. Moreover, since $G > 0$ except on $\{(x, v(x) + j) \mid x \in \mathbb{T}^n, j \in \mathbb{Z}\}$, it readily follows that $\mathcal{M}_0(F + \varepsilon G) = \{v + j \mid j \in \mathbb{Z}\}$ and the proposition follows.

Proposition 2.2 showed that $\mathcal{M}_0 = \mathcal{M}_0(\ell)$, i.e. by seeking solutions of (PDE) with integer periods other than 1, nothing new is obtained. In the same vein, the other results of Chapter 2 and this section required

$$u(x + e_i) = u(x), \quad 2 \leq i \leq n, \tag{3.57}$$

but that the period was 1 played no role. Thus with inessential changes, these results are also true if (3.57) is replaced by

$$u(x + \ell_i e_i) = u(x), \quad 2 \leq i \leq n, \tag{3.58}$$

where $\ell_i \in \mathbb{N}$. In particular, Theorem 3.2 provides the set $\mathcal{M}_1(\ell)$ of solutions of (PDE) heteroclinic in x_1, satisfying (3.58), and minimizing $J_1(\ell, u)$ over $\Gamma_1(\ell)$, where $\ell = (\ell_2, \ldots, \ell_n)$ and $J_1(\ell, u)$ and $\Gamma_1(\ell)$ are the natural extensions of J_1 and Γ_1 to this setting. Moreover the following version of Proposition 2.2, which will be required in Chapter 4, shows that no new solutions are obtained in this fashion.

Proposition 3.59. $\mathcal{M}_1(\ell) = \mathcal{M}_1$ *and* $c_1(\ell) \equiv \inf_{u \in \Gamma_1(\ell)} J_1(\ell, u) = \left(\prod_2^n \ell_i\right) c_1$.

Proof. Using that $\mathcal{M}_1(\ell)$ is ordered and $u(x + e_i) \in \mathcal{M}_1(\ell)$, $2 \leq i \leq n$, the proof follows exactly as in Proposition 2.2.

To conclude this section, the relationship between the solutions of (PDE) that have been constructed here, namely $\mathcal{M}_0, \mathcal{M}_1(v_0, w_0)$, and $\mathcal{M}_1(w_0, v_0)$, and solutions of (PDE) that are minimal and WSI will be explored. As was mentioned in Chapter 1, when $(*)_0$ holds, Bangert found solutions of (PDE) of this type that were heteroclinic in x_1 from v_0 to w_0 and periodic in $x_2, \ldots, x_n$ as well as solutions heteroclinic from w_0 to v_0. Among other things, the next theorem shows that Bangert's solutions precisely constitute $\mathcal{M}_1(v_0, w_0) \cup \mathcal{M}_1(w_0, v_0)$.

Theorem 3.60. *Let F satisfy (F_1)–(F_2).*

1^o If $u \in \mathcal{M}_0$ or if $(*)_0$ holds and $u \in \mathcal{M}_1(v_0, w_0) \cup \mathcal{M}_1(w_0, v_0)$, then u is minimal and WSI.

2^o *If u is a solution of (PDE) with $u(x + e_i) = u(x)$, $2 \le i \le n$, with rotation vector* 0, *and is minimal and WSI, then $u \in \mathcal{M}_0$ or $(*)_0$ holds and $u \in \mathcal{M}_1(v_0, w_0) \cup \mathcal{M}_1(w_0, v_0)$ for some adjacent pair $v_0, w_0 \in \mathcal{M}_0$.*

Proof. 1^o If $u \in \mathcal{M}_0 \cup \mathcal{M}_1(v_0, w_0) \cup \mathcal{M}_1(w_0, v_0)$, by 2^o(c) of Theorem 3.2, it is clear that u is WSI. To see that u is also minimal, let Ω be any bounded domain in $\mathbb{R}^n$ with a smooth boundary. Suppose $u \in \mathcal{M}_0$. By shifting the origin to some appropriate $j \in \mathbb{Z}^n$, it can be assumed that $\Omega \subset [0, \ell_1] \times \cdots \times [0, \ell_n]$ for some $\ell \in \mathbb{N}^n$. Then in the notation of Chapter 2,

$$\inf_{\mathcal{S}} \int_{\Omega} L(f)dx \tag{3.61}$$

exists, where

$$\mathcal{S} = \{f \in \Gamma_0(\ell) | f = u \quad \text{in} \quad ([0, \ell_1] \times \cdots \times [0, \ell_n]) \backslash \Omega\}.$$

Moreover, the inf in (3.61) is achieved by some $g \in \mathcal{S}$. If u is not a minimizer, then via Proposition 2.2, $J_0^\ell(g) < J_0^\ell(u) = c_0(\ell)$, a contradiction. Thus u is minimal. Similarly, using Proposition 3.59 shows that if $u \in \mathcal{M}_1(v_0, w_1) \cup \mathcal{M}_1(w_0, v_1)$, then u is minimal and 1^o holds.

To prove 2^o, the following technical result is needed.

Lemma 3.62. *If $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ is minimal, then for any $\varphi \in W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ with compact support,*

$$\int_{\mathbb{R} \times \mathbb{T}^{n-1}} (L(u + \varphi) - L(u))dx \ge 0. \tag{3.63}$$

Proof. Let $\widehat{x} = (x_2, \ldots, x_n)$ and $\ell \in \mathbb{N}$. For $s \in \mathbb{R}$, let $\theta_\ell(s)$ be a C^1 function such that $\theta_\ell = 1$ on $|s| \le \ell$; $\theta_\ell = 0$ if $|s| \ge \ell + 1 \ge 1$, and $0 \le \theta_\ell \le 1$. Since u is minimal,

$$0 \le \int_{\mathbb{R}^n} (L(u + \theta_\ell(|\hat{x}|)\varphi) - L(u))dx. \tag{3.64}$$

Suppose the support of φ lies in $[p, q + 1] \times \mathbb{T}^{n-1}$ with $p, q \in \mathbb{Z}$. Then (3.64) can be rewritten as

$$\begin{aligned} 0 &\le \int_{[p,q+1]\times[-\ell-1,\ell+1]^{n-1}} (L(u + \theta_\ell\varphi) - L(u))dx \\ &= (2\ell)^{n-1} \int_{[p,q+1]\times[0,1]^{n-1}} (L(u + \varphi) - L(u))dx + \mathcal{R}_\ell(u, \varphi), \end{aligned} \tag{3.65}$$

where

$$\mathcal{R}_\ell(u, \varphi) = \int_{A_\ell} (L(u + \theta_\ell\varphi) - L(u))dx$$

and A_ℓ is the region

$$[p, q+1] \times ([-\ell-1, \ell+1]^{n-1} \backslash [-\ell, \ell]^{n-1}).$$

Since $u, \varphi \in W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$, for each of the $(q-p+1)[(2(\ell+1))^{n-1} - (2\ell)^{n-1}]$ unit cubes a_i in $\mathbb{R}^n$ that make up A_ℓ, we have an estimate of the form

$$\left| \int_{a_i} (L(u+\theta_\ell \varphi) - L(u)) dx \right| \le M \tag{3.66}$$

where M depends on u and φ but not ℓ. Therefore by (3.65)–(3.66),

$$0 \le (2\ell)^{n-1} \int_{\mathbb{R}\times\mathbb{T}^{n-1}} (L(u+\varphi) - L(u)) dx + M b \ell^{n-2}, \tag{3.67}$$

where b depends on n and φ. Hence dividing (3.67) by $(2\ell)^{n-1}$ and letting $\ell \to \infty$ yields (3.63).

Proof of 2º of Theorem 3.60. The proof here requires more work. Let u be a solution of (PDE) with rotation vector $\alpha = 0$ that is minimal and WSI. Since $\alpha = 0$, by Theorem 1.2, u is bounded. Therefore there are a smallest w and largest v in $\mathcal{M}_0$ such that $v \le u \le w$. If $v = w$, $u \in \mathcal{M}_0$. Thus suppose that $v < w$. Then as in the argument involving (2.5), $v(x) < u(x) < w(x)$ for all x. Since u is WSI, $\tau^1_{-1} u = u$, $\tau^1_{-1} u > u$, or $\tau^1_{-1} u < u$.

Suppose

$$\tau^1_{-1} u < u. \tag{3.68}$$

For $k \in \mathbb{Z}$, the sequence of functions $u_k = \tau^1_k u$ is bounded in $C^2(T_0)$. Since by (3.68),

$$u_{k+1} > u_k, \tag{3.69}$$

as $k \to \infty$, u_k converges in $C^2(T_0)$ to $\overline{u} \le w$. Similarly, as $k \to -\infty$, $u_k \to \underline{u} \ge v$. By (3.69),

$$\tau^1_{-1} \varphi = \varphi \tag{3.70}$$

for $\varphi \in \{\underline{u}, \overline{u}\}$. Thus $\underline{u}, \overline{u} \in \Gamma_0$.

We claim that $\varphi \in \mathcal{M}_0$ for $\varphi \in \{\underline{u}, \overline{u}\}$. For example, if $\varphi = \overline{u}$ and

$$J_0(\varphi) > c_0, \tag{3.71}$$

there is a $k_0 \in \mathbb{N}$ such that for $k \ge k_0$,

$$J_0(u_k) - c_0 \ge \frac{1}{2}(J_0(\overline{u}) - c_0) \equiv \gamma > 0. \tag{3.72}$$

Therefore for $q \geq p + 4 \geq k_0$,

$$J_{1;p,q}(u) \geq (q - p)\gamma. \tag{3.73}$$

Set

$$f_{p,q} = \begin{cases} (p + 2 - x_1)u + (x_1 - (p + 1))w, & p + 1 \leq x_1 \leq p + 2, \\ w, & p + 2 \leq x_1 \leq q - 2, \\ (q - 2 - x_1)w + (x_1 - (q - 1))u, & q - 2 \leq x_1 \leq q - 1, \\ u, & \text{otherwise.} \end{cases}$$

By Lemma 3.62,

$$0 \geq J_{1;p,q}(u) - J_{1;p,q}(f_{p,q}). \tag{3.74}$$

Then by (3.73)–(3.74),

$$\begin{aligned} 0 &\geq J_{1;p+1,q-1}(u) - J_{1,p+1}(f_{p,q}) - J_{1,q-2}(f_{p,q}) \\ &\geq (q - p - 2)\gamma - J_{1,p+1}(f_{p,q}) - J_{1,q-2}(f_{p,q}). \end{aligned} \tag{3.75}$$

Since for $\lambda \in [0, 1]$, $\lambda u_k + (1 - \lambda)w \to \lambda\overline{u} + (1 - \lambda)w$ as $k \to \infty$,

$$J_0(\lambda u_k + (1 - \lambda)w) \to J_0(\lambda\overline{u} + (t - \lambda)w).$$

Therefore the last two terms on the right in (3.75) are bounded. Hence (3.75) cannot hold for $q-p$ large. Thus $J_0(\overline{u}) = c_0$ and similarly $J_0(\underline{u}) = c_0$. It follows that $\underline{u} = v$, $\overline{u} = w$, and $u \in \Gamma_1(v, w)$. The argument of this paragraph also shows $\tau^1_{-1}u = u$ is not possible unless $v = w$.

Next it will be shown that

$$J_1(u) = c_1(v, w). \tag{3.76}$$

Therefore by Theorem 3.34, v and w are adjacent members of $\mathcal{M}_0$, so $(*)_0$ holds and $u \in \mathcal{M}_1(v, w)$. If (3.76) is false, since $u \in \Gamma_1(v, w)$,

$$J_1(u) > c_1(v, w). \tag{3.77}$$

(Note that the left-hand side of (3.77) may be infinite.) Choose $U \in \Gamma_1(v, w)$ such that for some $\sigma > 0$,

$$c_1 \leq J_1(U) < J_1(U) + \sigma < J_1(u). \tag{3.78}$$

Let $\kappa > 0$. Then there is a $q = q(\kappa) \in \mathbb{N}$ such that for $\varphi \in \{u, U\}$,

$$\begin{cases} \|\varphi - v\|_{W^{1,2}(T_i)} \leq \kappa, & i \leq -q, \\ \|\varphi - w\|_{W^{1,2}(T_i)} \leq \kappa, & i \geq q. \end{cases} \tag{3.79}$$

For $i \in \mathbb{Z}$ and $i \leq x_1 \leq i+1$, set

$$g_i = (i - x_1)U + (x_1 + 1 - i)u.$$

Thus for $\kappa = \kappa(\sigma)$ sufficiently small and $\varphi \in \{u, U, g_i, h_i\}$,

$$|J_{1,i}(\varphi) - c_0| \leq \sigma/6 \tag{3.80}$$

for $|i| \geq q(\kappa)$. Let $p \in \mathbb{N}$, $p > q$. For p sufficiently large,

$$J_{1,-p,p}(U) \leq J_1(U) + \sigma/6. \tag{3.81}$$

Set

$$\psi = \begin{cases} u, & x_1 \leq -p, \\ g_{-p}, & -p \leq x_1 \leq -p+1, \\ U, & -p+1 \leq x_1 \leq p-1, \\ g_p, & p-1 \leq x_1 \leq p, \\ u, & p \leq x_1. \end{cases}$$

Consider

$$\begin{aligned} &\int_{[-p,p]\times\mathbb{T}^{n-1}} (L(u) - L(U))dx \\ &\quad = \int_{[-p-1,p+1]\times\mathbb{T}^{n-1}} (L(u) - L(\psi))dx + J_{1,-p}(\psi) - J_{1,-p}(U) \\ &\quad\quad + J_{1,p-1}(\psi) - J_{1,p-1}(U). \end{aligned} \tag{3.82}$$

By Lemma 3.62, the first term on the right in (3.82) is ≤ 0, while by (3.80), each of the other terms on the right is $\leq \sigma/6$ in magnitude. On the other hand,

$$\begin{aligned} \int_{[-p,p]\times\mathbb{T}^{n-1}} (L(u) - L(U))dx &= J_{1;-p,p-1}(u) - J_{1;-p,p-1}(U) \\ &\geq J_{1,-p,p-1}(u) - J_1(U) - \sigma/6 \end{aligned} \tag{3.83}$$

via (3.81). If $J_1(u) = \infty$, the right-hand side of (3.83) goes to ∞, as $p \to \infty$ while if $J_1(u) < \infty$, the right-hand side of (3.83) exceeds $2\sigma/3$ for large p, contrary to (3.82).

The remaining case of $u < \tau^1_{-1}u$ is treated similarly, and Theorem 3.60 is proved.

Chapter 4
Heteroclinics in x_1 and x_2

In this section the results of Chapters 2–3 will be extended to the next level of complexity, providing solutions of (PDE) heteroclinic in both x_1 and x_2. To describe such solutions more precisely, suppose that $(*)_0$ holds and also $\mathcal{M}_1 = \mathcal{M}_1(v_0, w_0)$ has gaps, i.e.,

$$\text{there are adjacent } v_1, w_1 \in \mathcal{M}_1(v_0, w_0) \text{ with } v_1 < w_1. \qquad (*)_1$$

It will be shown using minimization arguments in the spirit of Chapters 2–3 that there is a solution of (PDE) heteroclinic in x_2 from v_1 to w_1 (and therefore heteroclinic in x_1 from v_0 to w_0) and also periodic in $x_3, \ldots, x_n$. This provides a variational characterization of the corresponding part of Bangert's work. In Chapter 5, it will be indicated how to get heteroclinics in $x_1, \ldots, x_i$ for any $i \leq n$. The general program and many of the technical details for this section are close to those of Chapters 2–3 and therefore we will be brief whenever possible, focusing on the additional features present here. The new difficulties are mainly due to the compact sets T_i of Chapters 2–3 being replaced here by unbounded regions $\mathbb{R} \times [i, i+1] \times \mathbb{T}^{n-2}$.

To begin, let $v, w \in \mathcal{M}_1$, $v < w$, and define

$$\widehat{\Gamma}_2 \equiv \widehat{\Gamma}_2(v, w) \equiv \{u \in W^{1,2}_{\text{loc}}(\mathbb{R}^2 \times \mathbb{T}^{n-2}) \mid v \leq u \leq w\}.$$

For $u \in \widehat{\Gamma}_2$ and $\ell, i \in \mathbb{Z}$,

$$\|\tau^2_{-i}\tau^1_{-\ell}u - v_0\|_{L^2([0,1]^2 \times \mathbb{T}^{n-2})} \leq \|w - v_0\|_{L^2(T_\ell)} \to 0, \quad \ell \to -\infty \tag{4.1}$$

and similarly

$$\|\tau^2_{-i}\tau^1_{-\ell}u - w_0\|_{L^2([0,1]^2 \times \mathbb{T}^{n-2})} \to 0, \quad \ell \to \infty. \tag{4.2}$$

Thus $\tau^2_{-i}u$ satisfies the asymptotic conditions required of members of Γ_1. However, $\tau^2_{-i}u$ is not periodic in x_2, so a priori, $J_1(\tau^2_{-i}u)$ is not defined. It will be shown next how to extend J_1 to $\tau^2_{-i}u$ for $u \in \widehat{\Gamma}_2$. For such u, define

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$$J_1(u) = \varliminf_{\substack{p\to-\infty\\ q\to\infty}} J_{1;p,q}(u). \tag{4.3}$$

We claim that $J_{1;p,q}(u)$ is bounded from below independently of $u \in \widehat{\Gamma}_2$ and p, q. Further observing that $\tau^{\ell}_{-i} : \widehat{\Gamma}_2 \to \widehat{\Gamma}_2$ for $i \in \mathbb{Z}$ and $\ell = 2, \ldots, n$, it then follows that the extension of J_1 can be carried out. To verify the claim, for $i \in \mathbb{Z}$, set $S_i = \mathbb{R} \times [i, i+1] \times \mathbb{T}^{n-2}$. Then as in (2.9),

$$\begin{aligned} J_{1;p,q}(u) = &\int_{S_0\cap\{p\le x_1\le q+1\}} \Big[\frac{1}{2}|\nabla(u-v)|^2 + \nabla(u-v)\cdot\nabla v \\ &+\frac{1}{2}|\nabla v|^2 + F(x,u) - F(x,v) + F(x,v)\Big]\, dx - (q+1-p)c_0 \\ =&J_{1;p,q}(v) + \int_{S_0\cap\{p\le x_1\le q+1\}} \Big[\frac{1}{2}\nabla(u-v)|^2 + \nabla(u-v)\cdot\nabla v \\ &+ (F(x,u) - F(x,v))\Big]\, dx. \end{aligned} \tag{4.4}$$

As $-p, q \to \infty$, $J_{1;p,q}(v) \to J_1(v) = c_1$. To analyze the remaining terms, note first that

$$\begin{aligned} \int_{S_0} \big|F(x,u) - F(x,v)\big| dx &\le \|F_u\|_{L^\infty(\mathbb{T}^{n+1})} \int_{S_0} (u-v)dx \\ &\le \|F_u\|_{L^\infty(\mathbb{T}^{n+1})} \int_{S_0} (w-v)dx. \end{aligned} \tag{4.5}$$

Since $v, w \in \mathcal{M}_1$, $w < \tau^1_{-j} v$ for some smallest $j > 0$. Therefore

$$\int_{S_0} (w-v)dx \le \int_{S_0} (\tau^1_{-j} v - v)dx \le j \int_{T_0} (w_0 - v_0)dx \le j. \tag{4.6}$$

Thus the integral on the left in (4.5) is finite, and as estimates like (4.5)–(4.6) show,

$$\int_{S_0} (F(x,u) - F(x,v))dx$$

differs from the corresponding term in (4.4) by the tail of a convergent integral. Hence it is the limit of the corresponding term in (4.4) as $-p, q \to \infty$. Next as in (2.11),

$$\begin{aligned} \int_{S_0\cap\{p\le x_1\le q+1\}} \nabla(u-v)\cdot\nabla v\, dx = &\int_{\partial(S_0\cap\{p\le x_1\le q+1\})} (u-v)\frac{\partial v}{\partial \nu} dS \\ &- \int_{S_0\cap\{p\le x_1\le q+1\}} (u-v)\Delta v\, dx. \end{aligned} \tag{4.7}$$

Since $\Delta v = F_u(x, v)$, the argument of (4.5)–(4.6) shows that

$$\int_{S_0} (u - v)\Delta v \, dx$$

exists, is bounded as in (4.5)–(4.6), and is the limit of the corresponding integral over $S_0 \cap \{p \leq x_1 \leq q + 1\}$. The boundary integral in (4.7) has contributions from $x_2 = 0, 1$ and from $x_1 = p, q + 1$. Each of the x_2 boundary integrals is bounded by

$$\left\| \frac{\partial v}{\partial x_2} \right\|_{L^\infty(\mathbb{R}\times\mathbb{T}^{n-1})} \int_{S_0\cap\{x_2=0\}} (w - v) dS \leq j \left\| \frac{\partial v}{\partial x_2} \right\|_{L^\infty(\mathbb{R}\times\mathbb{T}^{n-1})}$$

as in (4.6). The remaining two boundary integrals are bounded by

$$\left\| \frac{\partial v}{\partial x_1} \right\|_{L^\infty(\mathbb{R}\times\mathbb{T}^{n-1})} \int_{S_0\cap\{x_1=p \text{ or } q+1\}} (w - v) dS.$$

Since $\|\tau^1_{-p}\varphi - v_0\|_{C^2(T_0)} \to 0$ as $p \to -\infty$, and $\|\tau^1_{-q}\varphi - w_0\|_{C^2(T_0)} \to 0$ as $q \to \infty$, where $\varphi = v$ or w, these integrals go to 0 as $p \to -\infty$, $q \to \infty$. Therefore

$$\int_{S_0\cap\{p\leq x_1\leq q+1\}} \nabla(u - v) \cdot \nabla v \, dx$$

has a finite limit. Thus it has been verified that J_1 extends to $\widehat{\Gamma}_2$. Moreover, the above shows that

$$J_1(u) = \infty \quad \Leftrightarrow \quad \|\nabla(u - v)\|^2_{L^2(S_0)} = \infty, \tag{4.8}$$

and if $\|\nabla(u - v)\|^2_{L^2(S_0)} < \infty$, a variation of this argument implies

$$\begin{aligned} J_1(u) = c_1 &+ \frac{1}{2}\|\nabla(u - v)\|^2_{L^2(S_0)} + \int_{S_0} (F(x, u) - F(x, v))dx \\ &+ \int_{S_0\cap\{|x_1|<r\}} \nabla(u - v) \cdot \nabla v \, dx \\ &+ \int_{\partial(S_0\cap\{|x_1|\geq r\})} (u - v)\frac{\partial v}{\partial \nu} \, dS - \int_{S_0\cap\{|x_1|\geq r\}} (u - v)\Delta v \, dx, \end{aligned} \tag{4.9}$$

the latter two integrals bounded independently of r, with zero limits as $r \to \infty$. In particular, since $w \in \widehat{\Gamma}_2$ and $J_1(w) = c_1$, $\|\nabla(w - v)\|_{L^2(S_0)} < \infty$.

To find the type of solutions of (PDE) that we seek here, as in Chapter 2 a renormalized functional, $J_2(u)$, is required. It is defined in a similar fashion to J_1. For $u \in \widehat{\Gamma}_2$ and $i \in \mathbb{Z}$, set

$$J_{2,i}(u) \equiv J_1(\tau^2_{-i}u) - c_1 = J_1(u\big|_{S_i}) - c_1.$$

By the above remarks, $J_{2,i}$ is defined on $\widehat{\Gamma}_2$ for each $i \in \mathbb{Z}$, as is

$$J_{2;p,q}(u) = \sum_{p}^{q} J_{2,i}(u)$$

for $p \le q$ in $\mathbb{Z}$. To continue, an analogue of Proposition 2.8 is required.

Proposition 4.10. *Suppose that $u \in \widehat{\Gamma}_2(v,w)$ and $p,q \in \mathbb{Z}$. Then there is a constant $K_2 \ge 0$ depending on v and w but independent of p,q,u such that*

$$J_{2;p,q}(u) \ge -K_2.$$

Proof. If $\|\nabla(u-v)\|^2_{L^2(S_i)} = \infty$ for some i then $J_{2;p,q}(u) = \infty$ by (4.8); otherwise, $\|\nabla(u-v)\|^2_{L^2(S_i)} < \infty$ for all $i \in \mathbb{Z}$ and by (4.5)–(4.9), with $r = 0$,

$$\left| J_{2,i}(u) - \frac{1}{2}\|\nabla(u-v)\|^2_{L^2(S_i)} \right| \le M_2, \tag{4.11}$$

where M_2 is a constant independent of i. This proves the proposition for $q = p$, $p+1, p+2$ with any $K_2 \ge 3M_2$. Thus suppose $q > p+2$ and define χ as in (2.15) with x_1 replaced by x_2. By Proposition 3.59, $J_{2;p,q}(\chi) \ge 0$. Continuing as in (2.16)–(2.19) (using $|u-v| \le w_0 - v_0 \le 1$, and arguing as following (4.7) to handle the boundary integral term resulting from the "new" (2.17)) yields Proposition 4.10 for this case.

Proposition 4.10 permits us to define

$$J_2(u) = \varliminf_{\substack{p\to-\infty \\ q\to\infty}} J_{2;p,q}(u) \tag{4.12}$$

for $u \in \widehat{\Gamma}_2$. Note that (4.8) implies

$$\|\nabla(u-v)\|_{L^2(S_i)} = \infty \text{ for some } i \in \mathbb{Z} \quad \Rightarrow \quad J_2(u) = \infty. \tag{4.13}$$

Then as in Chapter 2, $J_2(u)$ provides an upper bound for $J_{2;p,q}(u)$:

Lemma 4.14. *If $u \in \widehat{\Gamma}_2$, $p,q \in \mathbb{Z}$ with $p \le q$, then*

$$J_{2;p,q}(u) \le J_2(u) + 2K_2. \tag{4.15}$$

Proof. As in Lemma 2.22.

Now the class of functions in which the new heteroclinic solutions of (PDE) will be obtained is

$$\Gamma_2 \equiv \Gamma_2(v,w) \equiv \{u \in \widehat{\Gamma}_2 \mid \|u-v\|_{L^2(S_i)} \to 0, i \to -\infty, \text{ and } \\ \|u-w\|_{L^2(S_i)} \to 0, i \to \infty\}.$$

As in Chapter 2, J_2 has nicer properties on Γ_2:

Proposition 4.16. *If* $u \in \Gamma_2$ *and* $J_2(u) < \infty$*, then*

$$J_{2,i}(u) \to 0, \quad |i| \to \infty, \tag{4.17}$$

$$\|\tau_{-i}^2 u - v\|_{W^{1,2}(S_0)} \to 0, \; i \to -\infty, \tag{4.18}$$

$$\|\tau_{-i}^2 u - w\|_{W^{1,2}(S_0)} \to 0, \; i \to \infty, \tag{4.19}$$

$$J_2(u) = \lim_{\substack{p \to -\infty \\ q \to \infty}} J_{2;p,q}(u). \tag{4.20}$$

Proof. The proof follows the same lines as that of Proposition 2.24. However, since the compact set T_0 is replaced by the unbounded set S_0 here, some modifications are necessary. Replacing (2.29) by

$$\|\tau_{-i}^2 u - v\|_{L^2(S_0)}^2 \le \|w - v\|_{L^\infty(\mathbb{R}^n)} \int_{S_0} (w - v)dx \le j\,\|w - v\|_{L^\infty(\mathbb{R}^n)} \tag{4.21}$$

(where (4.6) was used) and arguing as in the proof of Proposition 2.24 shows that $\tau_{-i}^2 u - v \to 0$ weakly in $W^{1,2}(S_0)$ and strongly in $L^2(S_0)$ as $i \to -\infty$. This implies

$$\int_{S_0} (F(x, \tau_{-i}^2 u) - F(x, v))dx \to 0 \tag{4.22}$$

as $i \to -\infty$. Indeed,

$$\left| \int_{S_0} (F(x, \tau_{-i}^2 u) - F(x, v))dx \right| \le \|F_u\|_{L^\infty(\mathbb{T}^{n+1})} \int_{S_0} (\tau_{-i}^2 u - v)dx, \tag{4.23}$$

and for any $r > 0$,

$$\begin{aligned} \int_{S_0} (\tau_{-i}^2 u - v)dx &= \int_{S_0 \cap \{|x_1| > r\}} (\tau_{-i}^2 u - v)dx + \int_{S_0 \cap \{|x_1| \le r\}} (\tau_{-i}^2 u - v)dx \\ &\le \int_{S_0 \cap \{|x_1| > r\}} (w - v)dx + \int_{S_0 \cap \{x_1| \le r\}} (\tau_{-i}^2 u - v)dx. \end{aligned} \tag{4.24}$$

As $r \to \infty$, the first term on the right approaches 0, while for any r, the second term approaches 0 as $i \to -\infty$ via the $L^2(S_0)$ convergence of $\tau_{-i}^2 u$ to v. Thus (4.22) follows, and arguing as in the proof of Proposition 2.24 and above, (4.9) shows that

$$\underline{\lim}_{i \to -\infty} J_{2,i}(u) = \underline{\lim}_{i \to -\infty} \frac{1}{2} \|\nabla(\tau_{-i}^2 u - v)\|_{L^2(S_0)}^2 = 0. \tag{4.25}$$

A similar result for $i \to \infty$ holds, and this gives (4.17) with lim replaced by $\underline{\lim}$ and (4.18)–(4.19) along a subsequence. The proof now continues and concludes in a manner similar to that of Proposition 2.24. The inequality

$$\|\nabla(\chi - v)\|^2_{L^2(S_p)} \le 2\|\nabla(u - v)\|^2_{W^{1,2}(S_p)}$$

is used, and continuity of $J_{1,i}$ is replaced by (4.9). To show that the left-hand side of (4.9) is close to c_1, one takes r large enough so the last two terms are small, then takes $\|\nabla(u - v)\|^2_{W^{1,2}(S_0)}$ small enough that the remaining terms are small. The term involving F is estimated as in the proof of (4.22).

Next it is useful to show that $J_{2,i}$ is weakly lower semicontinuous. The corresponding result for $J_{1,i}$ was trivial.

Lemma 4.26. *Suppose $i \in \mathbb{Z}$ and $\mathcal{Y} \subset \widehat{\Gamma}_2$ with $J_{2,i}(u) < \infty$ for all $u \in \mathcal{Y}$. Then $J_{2,i}$ is weakly lower semicontinuous (with respect to $\| \quad \|_{W^{1,2}(S_i)}$) on $\mathcal{Y}$.*

Proof. Let $(u_k) \subset \mathcal{Y}, u \in \mathcal{Y}$, and $u_k - u \to 0$ weakly in $W^{1,2}(S_i)$. By (4.9) with $r = 0$,

$$J_{2,i}(u_k) = \frac{1}{2}\|\nabla(u_k - v)\|^2_{L^2(S_i)} + \int_{S_i}(F(x,u_k) - F(x,v))dx$$
$$+ \int_{\partial S_i}(u_k - v)\frac{\partial v}{\partial \nu}\, dS - \int_{S_i}(u_k - v)\Delta v\, dx. \tag{4.27}$$

The argument of (4.22)–(4.24) shows that

$$\int_{S_i}(F(x,u_k) - F(x,u))dx \to 0 \quad \text{as } k \to \infty.$$

A similar argument handles the last two terms on the left-hand side of (4.27). Therefore by the weak lower semicontinuity of $\| \cdot \|_{W^{1,2}(S_i)}$,

$$\underline{\lim}_{k\to\infty} J_{2,i}(u_k) \ge \frac{1}{2}\|\nabla(u - v)\|^2_{L^2(S_i)} + \int_{S_i}(F(x,u) - F(x,v))dx$$
$$+ \int_{\partial S_i}(u - v)\frac{\partial v}{\partial \nu}\, dS - \int_{S_i}(u - v)\Delta v\, dx = J_{2,i}(u). \tag{4.28}$$

The next result is a further compactness property of J_2 corresponding to Proposition 2.50.

Proposition 4.29. *Let $\mathcal{Y} \in \widehat{\Gamma}_2(v, w)$ with the property*

(Y_1^2) if $u \in \mathcal{Y}$ and $\chi_R \in \widehat{\Gamma}_2$ with $\chi_R(x) = u(x)$ for $|x_2| \ge R$, then $\chi_R \in \mathcal{Y}$ for all large R.

Define

$$c(\mathcal{Y}) = \inf_{u\in\mathcal{Y}} J_2(u). \tag{4.30}$$

If $c(\mathcal{Y}) < \infty$ and (u_k) is a minimizing sequence for (4.30), then there is a $U \in \widehat{\Gamma}_2$ such that along a subsequence, $u_k \to U$ in $W^{1,2}(S_i)$ for all $i \in \mathbb{Z}$.

Proof. Let (u_k) be a minimizing sequence for (4.30). By (4.9), (4.15), and arguments as in (4.5)–(4.8), $(u_k - v)$ is bounded in $W^{1,2}(S_i)$ independently of i for all $i \in \mathbb{Z}$. Therefore there is a $U \in \widehat{\Gamma}_2$ such that $u_k - v$ goes to $U - v$ weakly in $W^{1,2}(S_i)$ for each i along a subsequence that can be taken to be the entire sequence. Thus the weak lower semicontinuity of $\|\cdot\|_{W^{1,2}(S_i)}$ implies that $\|\nabla(U-v)\|_{L^2(S_i)}$ is bounded in $W^{1,2}(S_i)$ independently of i. Since $u_k \to U$ in $L^2_{\text{loc}}(\mathbb{R}^2 \times \mathbb{T}^{n-2})$, a slight variant of (4.24) shows that $u_k - U \to 0$ in $L^2(S_i)$ for each $i \in \mathbb{Z}$. Define

$$\delta_i = \underline{\lim}_{s\to\infty} J_{2,i}(u_s) - J_{2,i}(U). \tag{4.31}$$

By (4.9) with $r = 0$, estimating terms as in the proof of Lemma 4.26,

$$\delta_i = \frac{1}{2} \underline{\lim}_{s\to\infty} (\|\nabla(u_s - v)\|^2_{L^2(S_i)} - \|\nabla(U - v)\|^2_{L^2(S_i)}). \tag{4.32}$$

Since

$$\|\nabla(u_s - U)\|^2_{L^2(S_i)} = \|\nabla(u_s - v)\|^2_{L^2(S_i)} + \|\nabla(U - v)\|^2_{L^2(S_i)} - 2\int_{S_i} \nabla(u_s - v)\cdot\nabla(U - v)dx,$$

$$\underline{\lim}_{s\to\infty} \|\nabla u_s - U)\|^2_{L^2(S_i)} = \underline{\lim}_{s\to\infty} \|\nabla(u_s - v)\|^2_{L^2(S_i)} - \|\nabla(U - v)\|^2_{L^2(S_i)}. \tag{4.33}$$

Combining (4.32)–(4.33) gives

$$2\delta_i = \underline{\lim}_{s\to\infty} \|\nabla(u_s - U)\|^2_{L^2(S_i)}. \tag{4.34}$$

Now slightly modifying (2.59)–(2.62) completes the proof of Proposition 4.29, with standard arguments involving (4.9) implying the analogue of (2.60).

The regularity result Proposition 2.64 readily carries over to this section:

Proposition 4.35. *Under the hypotheses of Proposition 4.29, suppose*

(Y_2^2) there is a minimizing sequence (u_k) for (4.30) such that for some $r \in \left(0, \frac{1}{2}\right)$, some $z \in \mathbb{R}$, all smooth φ with support in $B_r(z)$, and associated $t_0(\varphi) > 0$,

$$c(\mathcal{Y}) \le J_2(u_k + t\varphi) + \delta_k \tag{4.36}$$

for all $|t| \le t_0(\varphi)$, where $\delta_k = \delta_k(\varphi) \to 0$ as $k \to \infty$.

Then the weak limit U of u_k satisfies (PDE) in $B_r(z)$.

Proof. As earlier with appropriate changes in notation.

As a final preliminary, for $v \in \mathcal{M}_1(v_0, w_0)$, set

$$\Gamma_2(v) = \{u \in \widehat{\Gamma}_2(\tau_1^1 v, \tau_{-1}^1 v) \mid \|\tau_{-i}^2 u - v\|_{L^2(S_i)} \to 0 \text{ as } |i| \to \infty\}.$$

Define

$$c_2(v) = \inf_{u \in \Gamma_2(v)} J_2(u) \tag{4.37}$$

and set

$$\mathcal{M}_2(v) = \{u \in \Gamma_2(v) \mid J_2(u) = c_2(v)\}.$$

Then we have:

Theorem 4.38. *If F satisfies (F_1)–(F_2) and $(*)_0$ holds, then $c_2(v) = 0$ and $\mathcal{M}_2(v) = \{v\}$.*

Proof. Following the proof of Theorem 2.72 (with the natural changes due to the current setting) until Remark 2.77 shows that $c_2(v) = 0$. The analogue of Remark 2.77 here is that $\tau_1^1 v \le u \le \tau_{-1}^1 v$ can be replaced by $\tau_j^1 v \le u \le \tau_{-j}^1 v$ for any $j \in \mathbb{N}$. The proof then continues and concludes as earlier.

Now the main existence result of this section can be stated. To formulate it, set

$$c_2 = c_2(v_1, w_1) = \inf_{u \in \Gamma_2(v_1, w_1)} J_2(u). \tag{4.39}$$

Theorem 4.40. *If F satisfies (F_1)–(F_2) and $(*)_i$ holds, $i = 0, 1$, then*

1° There is a $U_2 \in \Gamma_2$ such that $J_2(U_2) = c_2$, i.e., $\mathcal{M}_2 \equiv \mathcal{M}_2(v_1, w_1) \equiv \{u \in \Gamma_2(v_1, w_1) \mid J_2(u) = c_2\} \neq \emptyset$.
2° Any $U \in \mathcal{M}_2$ satisfies

(a) U is a solution of (PDE),
(b) $\|U - v_1\|_{C^2(S_i)} \to 0$, $i \to -\infty$,
$\|U - w_1\|_{C^2(S_i)} \to 0$, $i \to \infty$,
i.e., U is heteroclinic in x_2 from v_1 to w_1,
(c) $v_1 < U < \tau_{-1}^2 U < w_1$ and $U < \tau_{-1}^1 U$.

3° $\mathcal{M}_2$ is an ordered set.

Proof. Proceed as in Chapter 3, changing the normalization to

$$\int_{[0,1]\times[i,i+1]\times\mathbb{T}^{n-2}} u_k \, dx \le \frac{1}{2}\int_{T_0} (v_1 + w_1)\, dx \le \int_{T_0} u_k \, dx \tag{4.41}$$

for all $i \in \mathbb{Z}, i < 0$, and for all $k \in \mathbb{N}$ to get U_2 satisfying the modified versions of (3.5). Moreover as, e.g., in the proof of Proposition 4.29, $\|\nabla(U_2 - v)\|_{L^2(S_i)} < \infty$ for all $i \in \mathbb{Z}$ and the analogues of (3.6)–(3.7) hold. In particular, $J_2(U_2) < \infty$. Now follow (A)–(D) as earlier with some small modifications, e.g., in (A), we require $\tau_2^1 w_1 \le \tau_1^1 v_1 \le u_k + t\varphi \le \tau_{-1}^1 w_1$ for $|t| \le t_0(\varphi)$, and in (B), (3.13) now becomes

$$U_2 \le \tau_{-1}^2 U_2 \tag{4.42}$$

and $U_2 \in \widehat{\Gamma}_2(v_1, w_1) \setminus \{v_1, w_1\}$, so an analogue of Corollary 2.49 shows that $U_2 \in \Gamma_2(v_1, w_1)$. Continuing in this fashion, the argument of Chapter 3 yields all of Theorem 4.40 except for

$$U < \tau_{-1}^1 U \tag{4.43}$$

whenever $U \in \mathcal{M}_2$.

To verify (4.43), a slight variant of the argument used to show that (4.42) holds will be used. Set $\Phi = \max(U, \tau_{-1}^1 U)$ and $\Psi = \min(U, \tau_{-1}^1 U)$. We claim that

$$\Phi \in \Gamma_2(\tau_{-1}^1 v_1, \tau_{-1}^1 w_1) \tag{4.44}$$

and

$$\Psi \in \Gamma_2(v_1, w_1). \tag{4.45}$$

If so, by earlier arguments,

$$J_2(\Phi) + J_2(\Psi) = J_2(U) + J_2(\tau_{-1}^1 U) = c_2(v_1, w_1) + c_2(\tau_{-1}^1 v_1, \tau_{-1}^1 w_1). \tag{4.46}$$

(Actually the two numbers on the right are equal.) Therefore by (4.44)–(4.46), $J_2(\Phi) = c_2(\tau_{-1}^1 v_1, \tau_{-1}^1 w_1)$ and $J_2(\Psi) = c_2(v_1, w_1)$. Hence by 2^o(a) of Theorem 4.40, Φ and Ψ are solutions of (PDE) with $\Phi \ge \Psi$. By the maximum principle argument of (2.5), either (i) $\Phi \equiv \Psi$, or (ii) $\Phi > \Psi$ on $\mathbb{R}^n$. If (i) holds, $U \equiv \tau_{-1}^1 U$. so U is 1-periodic in x_1. But then the requirement that $v_1 < U < w_1$ fails. Therefore (ii) occurs, so (a) $U > \tau_{-1}^1 U$ or (b) $U < \tau_{-1}^1 U$. If (a),

$$w_0 > w_1 > U \ge \lim_{j \to \infty} \tau_{-j}^1 U = w_0,$$

a contradiction. Thus (b), i.e. (4.43), is valid.

It remains to check that (4.44)–(4.45) hold. The arguments are the same for each inclusion, so (4.45) will be verified. Since $v_1 < U$ and $v_1 < \tau_{-1}^1 v_1 < \tau_{-1}^1 U$, $v_1 < \Psi \le U < w_1$. Therefore $\Psi \in \widehat{\Gamma}_2$. To check the asymptotic requirements of Γ_2, note first that

$$\|\Psi - v_1\|_{L^2(S_i)} \le \|U - v_1\|_{L^2(S_i)} \to 0, \quad i \to -\infty.$$

Next observe that

$$\int_{S_i} |\Psi - w_1|^2 dx = \int_{S_i \cap \{|x_1| \ge r\}} |\Psi - w_1|^2 dx + \int_{S_i \cap \{|x_1| \le r\}} |\Psi - w_1|^2 dx. \tag{4.47}$$

As in earlier arguments,

$$\int_{S_i \cap \{|x_1|>r\}} |\Psi - w_1|^2 dx \le \int_{S_0 \cap \{|x_1|>r\}} (\tau_{-1}^1 v_1 - v_1) dx, \tag{4.48}$$

and the right-hand side of (4.48) is the tail of a convergent integral. Therefore it goes to 0 as $r \to \infty$. Since $\tau_{-i}^2 U \to w_1$ and $\tau_{-1}^1 \tau_{-i}^2 U \to \tau_{-1}^1 w_1 > w_1$ as $i \to \infty$, convergence being in $C^2_{\text{loc}}(S_0)$,

$$\int_{S_i \cap \{|x_1| \le r\}} |\Psi - w_1|^2 dx \to 0, \quad \text{as } i \to \infty. \tag{4.49}$$

Combining (4.47)–(4.49) shows that

$$\|\Psi - w_1\|_{L^2(S_i)} \to 0, \quad i \to \infty,$$

and Theorem 4.40 is proved.

Next as in Theorem 3.34 we have:

Theorem 4.50. *Suppose F satisfies (F_1)–(F_2), $(*)_0$ holds, and $v, w \in \mathcal{M}_1(v_0, w_0)$ with $v \ne w$. Then $\mathcal{M}_2(v, w) \ne \emptyset$ iff v and w are adjacent members of $\mathcal{M}_1(v_0, w_0)$.*

Proof. The proof is the same as that of Theorem 3.34 with some obvious changes in notation.

Remark 4.51. It is straightforward to show that Proposition 3.59 carries over to the current setting.

The analogues of Propositions 3.42, 3.56 and Theorem 3.60 will be given next. To formulate a version of Proposition 3.42 for the current setting, suppose $(*)_0$ holds for F with a gap pair $v_0(F)$, $w_0(F)$. Then by Remark 3.55, for any $\overline{F}$ near F, there is a unique associated gap pair $v_0(\overline{F})$, $w_0(\overline{F})$ for $\overline{F}$. Suppose $(*)_1$ also holds for F and $v_1(F)$, $w_1(F)$ is any associated gap pair for $(*)_1$ for F (with $v_0(F) < v_1(F) \le w_1(F) < v_0(F)$). Then we expect a corresponding gap pair $v_1(\overline{F})$, $w_1(\overline{F})$ for $\overline{F}$. The next result is the first step in showing that this is the case.

Proposition 4.52. *Let F satisfy (F_1)–(F_2), $(*)_0$, and $(*)_1$. Then there is an $\epsilon > 0$ such that if (3.43) is satisfied, $(*)_1$ holds for $\overline{F}$. Moreover, suppose v_1, w_1 is a gap pair for F for $(*)_1$ and*

$$\alpha_1 = \int_{[0,1]^2 \times \mathbb{T}^{n-2}} v_1 \, dx; \qquad \beta_1 = \int_{[0,1]^2 \times \mathbb{T}^{n-2}} w_1 \, dx.$$

Then there is an $\epsilon_2 = \epsilon_2(F, \delta) > 0$ such that (3.43) with ϵ_2 implies

$$\int_{[0,1]^2 \times \mathbb{T}^{n-2}} u \, dx \notin (\alpha_1 + \delta, \beta_1 - \delta) \tag{4.53}$$

for all $u \in \mathcal{M}_1(v_0(\overline{F}), w_0(\overline{F}))$.

Proof. As in the proof of Proposition 3.42, if (4.53) is false, there are a $\delta \in (0, (\beta_1 - \alpha_1)/2)$ and a sequence (F_k) satisfying (F_1)–(F_2) and (3.45) with $u_k \in \mathcal{M}_1(v_0(F_k), w_0(F_k))$ such that

$$\int_{[0,1]^2 \times \mathbb{T}^{n-2}} u_k \, dx \in (\alpha_1 + \delta, \beta_1 - \delta). \tag{4.54}$$

Since $v_0(F_k) \le u_k \le w_0(F_k)$ and by Remark 3.55, $v_0(F_k), w_0(F_k)$ are near $v_0(F), w_0(F)$, it follows that (u_k) are bounded in $L^\infty(\mathbb{R} \times \mathbb{T}^{n-1})$. Therefore by the L^p_{loc} elliptic theory and estimates like (3.52), the functions u_k are bounded in $C^{1,\alpha}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ for any $\alpha \in (0,1)$. Passing to a limit as in Proposition 3.42 yields a solution u of (PDE) for F with

$$\int_{[0,1]^2 \times \mathbb{T}^{n-2}} u \, dx \in [\alpha_1 + \delta, \beta_1 - \delta]. \tag{4.55}$$

The functions u_k are minimal, so u is also minimal. Likewise $\tau^1_{-1} u_k > u_k$ implies

$$\tau^1_{-1} u \ge u, \tag{4.56}$$

and by the maximum principle, there is never equality in (4.56) unless u is 1-periodic in x_1. In either event, u is also WSI. Consequently, by 2^o of Theorem 3.60, $u \in \mathcal{M}_0$ or $u \in \mathcal{M}_1(v_0(F), w_0(F))$. But either of these possibilities is contrary to (4.55). Thus (4.53) must hold, and Proposition 4.52 is proved.

Remark 4.57. by Proposition 4.52, the remarks immediately preceding it, and Remark 3.55, for ϵ small in (3.43) there is a unique gap pair $v_1(\overline{F}), w_1(\overline{F})$ near $v_1(F), w_1(F)$.

The next result is a version of Proposition 3.56 for the current setting.

Theorem 4.58. *Suppose F satisfies (F_1)–(F_2). Then for any $\epsilon > 0$, there is a G satisfying (F_1)–(F_2) with*

1^o $\|G - F\|_{L^\infty(\mathbb{T}^{n+1})} + \|G_u - F_u\|_{L^\infty(\mathbb{T}^{n+1})} \le \epsilon$.
2^o $(*)_0$, $(*)_1$ *hold for* G.
3^o $\mathcal{M}_0(G) = \{v + j \mid j \in \mathbb{Z}\}$ *for some prescribed* $v \in \mathcal{M}_0(F)$.
4^o If $\mathcal{M}_1(v, v+1, G)$ *is the set of minimizers given by Theorem 3.2,* $\mathcal{M}_1(v, v+1, G) = \{\tau^1_{-k} U \mid k \in \mathbb{Z}\}$ *for some* $U \in \mathcal{M}_1(v, v+1, G)$.

Proof. By Proposition 3.56, for any prescribed $v \in \mathcal{M}_0(F)$, there is a $G_1 \ge 0$ satisfying (F_1)–(F_2) and such that $F + \delta_1 G_1$ satisfies $(*)_0$ for any $\delta_1 > 0$ and $\mathcal{M}_0(F) = \mathcal{M}_0(F + \delta_1 G_1) = \{v + j \mid j \in \mathbb{Z}\}$. Therefore $c_0(F) = c_0(F + \delta_1 G_1)$.

Consider the family of functionals $J_1^{F+\delta_1 G_1}$ on $\Gamma_1(v, v+1)$. Note that

$$J_1^{F+\delta_1 G_1}(u) = J_1^F(u) + \delta_1 \int_{\mathbb{R} \times \mathbb{T}^{n-1}} G_1(x, u(x)) dx. \tag{4.59}$$

Denote the associated minimum on $\Gamma_1(v, v+1)$ by $c_1(v, v+1, F+\delta_1 G_1)$ or more simply by $c_1(F+\delta_1 G_1)$. Likewise let $\mathcal{M}_1(v, v+1, F+\delta_1 G_1)$ or $\mathcal{M}_1(F+\delta_1 G_1)$ denote the corresponding set of minimizers. By Theorem 3.2, $\mathcal{M}_1(F+\delta_1 G_1) \neq \emptyset$ for all $\delta_1 > 0$. Choose δ_1 so that

$$\delta_1(\|G_1\|_{L^\infty(\mathbb{T}^{n+1})} + \|G_{1u}\|_{L^\infty(\mathbb{T}^{n+1})}) \leq \epsilon/2. \tag{4.60}$$

If $(*)_1$ holds for $\mathcal{M}_1(F+\delta_1 G_1)$, we have 1^o–3^o via Proposition 4.52. If not, $\mathcal{M}_1(F+\delta_1 G_1)$ foliates

$$A \equiv \{(x, z) \mid x \in \mathbb{R} \times \mathbb{T}^{n-1}, v(x) \leq z \leq v(x)+1\}.$$

Choose any $U_1 \in \mathcal{M}_1(F+\delta_1 G_1)$. We will show there is a G_2 satisfying (F_1)–(F_2) such that $G_2 \geq 0$, $G_2(x, \varphi(x)) = 0$ for $x \in \mathbb{R} \times \mathbb{T}^{n-1}$ and $\varphi \in \{v, v+1\} \cup \{\tau^1_{-j} U_1 \mid j \in \mathbb{Z}\}$ and $G_2(x, z) > 0$ on A aside from the above set of $\{x, \varphi(x)\}$. For such a G_2, consider $G = F + \delta_1 G_1 + \delta_2 G_2$. Then $J_0^G(v) = c_0(F)$ and if $u \in \Gamma_0 \backslash \{v + j \mid j \in \mathbb{Z}\}$,

$$J_0^G(u) \geq J_0(u) > c_0(F).$$

Hence $c_0(F) = c_0(G)$. Similarly if $u \in \Gamma_1(v, v+1)$,

$$J_1^G(u) = J_1^{F+\delta_1 G_1}(u) + \delta_2 \int_{\mathbb{R}\times\mathbb{T}^{n-1}} G_2(x, u)dx \geq c_1(F+\delta_1 G_1). \tag{4.61}$$

Therefore (4.61) shows that

$$c_1(F+\delta_1 G_1) = c_1(G). \tag{4.62}$$

Moreover, if $u \in \mathcal{M}_1(G) \backslash \cup \{\tau^1_{-j} U_1 \mid j \in \mathbb{Z}\}$, then u is continuous and $\{(x, u(x)) \mid x \in \mathbb{R} \times \mathbb{T}^{n-1}\}$ contains points in A where G_2 is positive. Therefore $J_1^G(u) = c_1(G) > c_1(F+\delta_1 G_1)$, contrary to (4.62).

Finally, to construct G_2, it suffices to define it on

$$A = \{(x, z) \mid x \in [0, 1] \times \mathbb{T}^{n-1}, v(x) \leq z \leq v(x)+1\}$$

and extend it periodically to $\mathbb{R}^{n+1}$. Set $G_2 = 0$ on

$$\{(x, v(x)) \mid x \in [0, 1] \times \mathbb{T}^{n-1}\} \cup \{(x, \tau^1_{-j} U_1(x)) \mid j \in \mathbb{Z}, x \in [0, 1] \times \mathbb{T}^{n-1}\}.$$

For z between $\tau^1_{-j} U_1$ and $\tau^1_{-j-1} U_1$ define

$$G_2(x, z) = |z - U_1(x+j)|^4 |z - U_1(x+j+1)|^4.$$

Then G_2 is C^2 and positive in the desired set. Taking δ_2 sufficiently small, 2^o holds, and the proof of Theorem 4.58 is complete.

Lastly, the versions of Theorem 3.60 in the context of this section will be considered. Using Remark 4.51 and the proof of 1^o of Theorem 3.60 readily shows

Theorem 4.63. *Let F satisfy (F_1)–(F_2) and let $(*)_0$, $(*)_1$ hold. If $u \in \mathcal{M}_2(v_1, w_1)$ (or $\mathcal{M}_2(w_1, v_1)$), then u is minimal and WSI.*

The most natural extension of 2^o of Theorem 3.60 would assume that u is 1-periodic in $x_3, \dots, x_n$ and offer the earlier alternatives plus allow the possibility that a version of $(*)_1$ holds and $u \in \mathcal{M}_2(v_1, w_1)$ for some adjacent pair $v_1, w_1 \in \mathcal{M}_1$. Unfortunately, this is not true. In fact, there are many other possibilities for u. This point will be taken up in Chapter 5 where the additional cases will be discussed. For now a milder result will be proved.

Proposition 4.64. *Let F satisfy (F_1)–(F_2) and $(*)_0$, $(*)_1$ hold. If $U \in \Gamma_2(v_1, w_1)$ is minimal and WSI, then $U \in \mathcal{M}_2(v_1, w_1)$.*

Proof. Being minimal, U is a solution of (PDE).

Since $U \in \Gamma_2(v_1, w_1)$,

$$J_2(U) \geq c_2. \tag{4.65}$$

We claim that

$$J_2(U) = c_2. \tag{4.66}$$

If so, $U \in \mathcal{M}_2(v_1, w_1)$ and the proof is complete. To verify (4.66), an argument in the spirit of the analogous conclusion in the proof of Theorem 3.60 will be employed. First, in order to cut and paste, it must be shown that as $j \to \infty$,

$$\|U - w_1\|_{W^{1,2}(S_j)}, \|U - v_1\|_{W^{1,2}(S_{-j})} \to 0. \tag{4.67}$$

Indeed, both U and w_1 are solutions of (PDE). Setting $\Phi = U - w_1$, as in (2.5), Φ satisfies

$$-\Delta\Phi + A\Phi = 0, \tag{4.68}$$

where $\|A\|_{L^\infty(\mathbb{R}^n)} \leq \|F_{uu}\|_{L^\infty(\mathbb{T}^{n+1})}$. Choose $\eta \in C^1$ such that $|\eta| \leq 1$, $\eta = 1$ on $\bigcup_{i=-1}^{1} S_{j+i}$, $\eta = 0$ outside of $\bigcup_{i=-2}^{2} S_{j+i}$, and $|\nabla\eta| \leq 3$. Multiply (4.68) by $\eta^2\Phi$ and integrate by parts to get

$$0 = \int_{\bigcup_{i=-2}^{2} S_{j+i}} \left(\eta^2|\nabla\Phi|^2 + 2\eta\Phi\nabla\eta \cdot \nabla\Phi + A\eta^2\Phi^2\right) dx. \tag{4.69}$$

Consequently, for any $\epsilon > 0$,

$$\int_{\bigcup_{i=-2}^{2} S_{j+i}} \eta^2|\nabla\Phi|^2 dx \leq \int_{\bigcup_{i=-2}^{2} S_{j+i}} \left[\epsilon^2\eta^2|\nabla\Phi|^2 + \left(\frac{9}{\epsilon^2} + \|A\|_{L^\infty(\mathbb{R}^n)}\right)\Phi^2\right] dx. \tag{4.70}$$

Choosing $\epsilon^2 = \frac{1}{2}$ yields

$$\frac{1}{2}\int_{\bigcup_{i=-1}^{1} S_{j+i}} |\nabla\Phi|^2 dx \le \frac{1}{2}\int_{\bigcup_{i=-2}^{2} S_{j+i}} \eta^2|\nabla\Phi|^2 dx$$
$$\le \int_{\bigcup_{i=-2}^{2} S_{j+i}} (18 + \|A\|_{L^\infty(\mathbb{R}^n)})\Phi^2 dx,$$

i.e.,

$$\int_{\bigcup_{i=-1}^{1} S_{j+i}} |\nabla(U - w_1)|^2 dx \le 2(18 + \|A\|_{L^\infty(\mathbb{R}^2)}) \int_{\bigcup_{i=-2}^{2} S_{j+i}} |U - w_1|^2\, dx. \tag{4.71}$$

The right-hand side of (4.71) goes to 0 as j approaches infinity since $U \in \Gamma_2\,(v_1, w_1)$. This fact with (4.71) and its analogue for v_1 imply (4.67).

Now to prove (4.66), we slightly modify the corresponding argument in the proof of Theorem 3.60. If (4.66) is false,

$$J_2(U) > c_2. \tag{4.72}$$

Choose $\psi \in \Gamma_2(v_1, w_1)$ such that for some $\sigma > 0$,

$$c_2 \le J_2(\psi) < J_2(\psi) + \sigma < J_2(U). \tag{4.73}$$

By (4.67) and Proposition 4.16, for any $\kappa > 0$, there is a $q = q(\kappa) \in \mathbb{N}$ such that for $\varphi \in \{U, \psi\}$,

$$\begin{aligned} &\|\varphi - v_1\|_{W^{1,2}(S_i)} \le \kappa, \quad i \le -q, \\ &\|\varphi - w_1\|_{W^{1,2}(S_i)} \le \kappa, \quad i \ge q. \end{aligned} \tag{4.74}$$

For $i \in \mathbb{Z}$ and $x_2 \in [i, i + 1]$, set

$$\begin{aligned} G_i &= (x_2 - i)\psi + (i + 1 - x_2)U, \\ H_{i+1} &= (x_2 - i)U + (i + 1 - x_2)\psi. \end{aligned}$$

Thus for $\kappa = \kappa(\sigma)$ sufficiently small and $\varphi \in \{U, \psi, G_i, H_{i+1}\}$,

$$|J_{2,i}(\varphi)| \le \sigma/6 \tag{4.75}$$

for $|i| \ge q(\kappa)$. Let $p \in \mathbb{N}$, $p > 1$. For p sufficiently large,

$$J_{2;-p,p-1}(\psi) \le J_2(\psi) + \sigma/6. \tag{4.76}$$

Set

$$\Psi = \begin{cases} U, & x_2 \le -p, \\ G_{-p}, & -p \le x_2 \le -p+1, \\ \psi, & -p+1 \le x_2 \le p-1, \\ H_p, & p-1 \le x_2 \le p, \\ U, & p \le x_2. \end{cases}$$

Consider

$$\int_{\mathbb{R}\times[-p,p]\times\mathbb{T}^{n-2}} (L(U) - L(\psi))dx = \int_{\mathbb{R}\times[-p-1,p+1]\times\mathbb{T}^{n-2}} (L(U) - L(\Psi))dx$$
$$+ J_{2,-p}(\Psi) - J_{2,-p}(\psi) + J_{2,p-1}(\Psi) - J_{2,p-1}(\psi). \tag{4.77}$$

The first term on the right is ≤ 0, since U is minimal. By (4.75), each of the remaining terms on the right is $\le \sigma/6$ in magnitude. To estimate the left-hand side of (4.77), we write

$$\int_{\mathbb{R}\times[-p,p]\times\mathbb{T}^{n-2}} (L(U) - L(\psi))dx = \sum_{i=-p}^{p-1} \int_{S_i} (L(U) - L(\psi))dx$$
$$= \sum_{i=-p}^{p-1} (J_1(\tau^2_{-i} U) - J_1(\tau^2_{-i}\psi))$$
$$= J_{2;-p,p-1}(U) - J_{2:-p,p-1}(\psi) \ge J_{2;-p,p-1}(U) - J_2(\psi) - \sigma/6 \tag{4.78}$$

via (4.76). Thus if $J_2(U) = \infty$, by (4.78) the left-hand side of (4.77) $\to \infty$ as $p \to \infty$, while if $J_2(U) < \infty$, by (4.73), the left-hand side of (4.77) exceeds $2\sigma/3$. In either case, we have a contradiction and (4.66) is valid, completing the proof of Proposition 4.79.

To conclude this section, a result that is needed to obtain extensions of Proposition 4.52 will be presented. To set the stage, suppose (F_k), F satisfy (F_1)–(F_2) and F_k, F satisfy (3.43) with $\epsilon = \epsilon_k \to 0$ as $k \to \infty$. Suppose also that $(*)_0$, $(*)_1$ hold for F. Then by Remark 4.57, for any gap pair $v_1(F)$, $w_1(F)$ for $(*)_1$ for F, whenever k is large there is a unique gap pair $v_1(F_k)$, $w_1(F_k)$ for $(*)_1$ for F_k that is near $v_1(F)$, $w_1(F)$. Moreover, as $k \to \infty$, $v_1(F_k) \to v_1(F)$ and $w_1(F_k) \to w_1(F)$. Let $U_k \in \mathcal{M}_2(v_1(F_k), w_1(F_k))$ be a solution of (PDE) given by Theorem 4.40. Then we have:

Proposition 4.79. *Along a subsequence,*

$$U_k \to U \in \mathcal{M}_2(v_1(F), w_1(F)) \cup \{v_1(F), w_1(F)\},$$

convergence being in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^2 \times \mathbb{T}^{n-2})$.

Proof. Note that $v_1(F_k) \le U_k \le w_1(F_k)$. It follows as in the proof of Proposition 4.52 that U_k converges along a subsequence to U, a solution of (PDE) with $v_1(F) \le U \le w_1(F)$, and U is minimal and WSI. Moreover, if $v_1(F)(z) = U(z)$ for some z, then as earlier, $v_1(F) \equiv U$, and likewise for $w_1(F)$. Thus suppose that

$$v_1(F) < U < w_1(F). \tag{4.80}$$

Since U is WSI, the functions $\tau^2_{-j}U$ form a monotone increasing sequence in j. Hence as $j \to \infty$, $\tau^2_{-j}U|_{\mathbb{R}\times[0,1]\times\mathbb{T}^{n-2}}$ converges in $C^{2,\alpha}_{loc}$ to a solution u of (PDE) that is minimal and WSI and is 1-periodic in $x_2, \ldots, x_n$. By (4.80),

$$v_1(F) < u \le w_1(F). \tag{4.81}$$

Theorem 3.60 implies $u \in \mathcal{M}_0 \cup \mathcal{M}_1(v_0(F), w_0(F))$. Since

$$v_0(F) < v_1(F) < w_1(F) < w_0(F),$$

$u \in \mathcal{M}_1(v_0(F), w_0(F))$. But $v_1(F), w_1(F)$ is a gap pair for $(*)_1$, so $u = w_1(F)$. Similarly $\tau^2_{-j}U \to v_1(F)$ as $j \to -\infty$. Therefore $U \in \widehat{\Gamma}_2(v_1(F), w_1(F))$.

We claim that $U \in \Gamma_2(v_1(F), w_1(F))$. This requires showing that

$$\|U - v_1(F)\|_{L^2(S_{-j})}, \|U - w_1(F)\|_{L^2(S_j)} \to 0, \quad j \to \infty. \tag{4.82}$$

Since for any $R > 0$,

$$\begin{aligned}\|U - w_1(F)\|^2_{L^2(S_j)} &= \int_{S_0} |\tau^2_{-j}U - w_1(F)|^2 dx \\ &= \int_{\{|x_1|\le R\}\cap S_0} + \int_{\{|x_1|>R\}\cap S_0} \equiv P_1 + P_2\end{aligned}$$

and $\tau^2_{-j}U \to w_1(F)$ in L^∞_{loc} as $j \to \infty$, $P_1 \to 0$ as $j \to \infty$. As in (4.6) or (4.24),

$$\begin{aligned}P_2 \le &\int_{\{|x_1|>R\}\cap S_0} (w_1(F) - \tau^2_{-j}U)dx \le \int_{\{|x_1|>R\}\cap S_0} (\tau^1_{-1}v_1(F) - v_1(F))dx \\ = &-\int_{[R,R+1]\times\mathbb{T}^{n-1}} v_1(F)dx + \lim_{N\to\infty}\int_{[R+N,R+N+1]\times\mathbb{T}^{n-1}} v_1(F)dx \\ &+ \int_{[-R,-R+1]\times\mathbb{T}^{n-1}} v_1(F)dx - \lim_{N\to\infty}\int_{[-R-N,-R-N+1]\times\mathbb{T}^{n-1}} v_1(F)dx.\end{aligned}$$

As $R \to \infty$, $v_1(F) \to w_0(F)$ uniformly on $[R, R+1] \times \mathbb{T}^{n-1}$ and as $R \to -\infty$, $v_1(F) \to v_0(F)$ uniformly on $[R, R+1) \times \mathbb{T}^{n-1}$. Therefore $P_2 \to 0$ as $R \to \infty$. It follows that (4.82) holds.

Consequently, U satisfies the hypotheses of Proposition 4.64. Hence $U \in \mathcal{M}_2(v_1(F), w_1(F))$ and Proposition 4.79 is proved.

Chapter 5
More Basic Solutions

The purpose of this chapter is to extend the results of Chapters 2–4 in three ways. This will be carried out in Sections 5.1–5.3 that follow. In Section 5.1, we briefly indicate how to modify Theorems 3.2 and 4.40 to obtain more complex heteroclinic solutions of (PDE). These new solutions, U_k, $3 \le k \le n$, are higher-dimensional analogues of U_1 of Theorem 3.2 and U_2 of Theorem 4.40. By higher-dimensional we mean that U_k is periodic in $x_{k+1}, \dots, x_n$, lies between U_{k-1} and $\tau_{-1}^{k-1} U_{k-1}$, and is heteroclinic in x_k from U_{k-1} to $\tau_{-1}^{k-1} U_{k-1}$. All of these new solutions have rotation vector $\alpha = 0$.

Still keeping $\alpha = 0$, in Section 5.2 it will be shown that for $k = 1, \dots, n$, in addition to the solutions of Chapters 2–4 and Section 5.1, there is a further infinitude of basic heteroclinic solutions corresponding to each of the U_k's. For example, in the simplest case, for $1 \le i \le n$, let $\alpha_{ij} \in \mathbb{Z}$ be relatively prime, $1 \le j \le n$, and set $\omega_i = \sum_1^n \alpha_{ij} e_j$. Suppose that $\omega_1, \dots, \omega_n$ are orthogonal. Then there is a solution $U_1(\omega_1, \dots, \omega_n)(x)$ of (PDE) that is heteroclinic from v_0 to w_0 in the ω_1 direction and periodic in the ω_i direction, $2 \le i \le n$. When ω_1 is not a multiple of e_1, these solutions are all distinct from $U_1(e_1, \dots, e_n)$. Moreover, there are also solutions $U_2(\omega_1, \dots, \omega_n)(x)$ in the spirit of Chapter 4, etc.

Lastly in Section 5.3, it will be discussed how all of the results obtained up to this point carry over to $\alpha \in \mathbb{Q}^n \backslash \{0\}$.

To begin, some remarks about notation are in order. When $(*)_0$ holds, Theorem 3.2 and Remark 3.31 provide at least two solutions of (PDE) heteroclinic in x_1 and periodic in $x_2, \dots, x_n$. Namely, up to the phase shifts, τ_{-j}^1, there is a pair of solutions, each heteroclinic in x_1, one from v_0 to w_0, and the other from w_0 to v_0. This leads to two versions of $(*)_1$, one each for $\mathcal{M}_1(v_0, w_0)$ and $\mathcal{M}_1(w_0, v_0)$. Likewise, each version of $(*)_1$ and Theorem 4.40 then provide a pair of solutions of (PDE) heteroclinic in x_2 between gap pairs in $\mathcal{M}_1$. Hence there are four versions of $(*)_2$, and at step k, 2^k versions of $(*)_k$. For simplicity this section will deal with the version of $(*)_i$ for which $U \in \mathcal{M}_k(v_{k-1}, w_{k-1})$ implies $\tau_{-1}^i U > U$, $1 \le i \le k$. The remaining cases are treated in the same way.

P.H. Rabinowitz and E.W. Stredulinsky, *Extensions of Moser–Bangert Theory*, Progress in Nonlinear Differential Equations and Their Applications 81,
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5.1 Higher-Dimensional Heteroclinics

Suppose the theory of Chapters 2–4 has been extended to level $\ell < n$. Using the notation just explained, to obtain results for level $\ell + 1$, assume

there is a gap in $\mathcal{M}_\ell \equiv \mathcal{M}_\ell(v_{\ell-1}, w_{\ell-1})$ given by adjacent $v_\ell, w_\ell \in \mathcal{M}_\ell$ with $v_\ell < w_\ell$. $(*)_\ell$

For $v, w \in \mathcal{M}_\ell$ with $v < w$ define

$$\widehat{\Gamma}_{\ell+1} = \widehat{\Gamma}_{\ell+1}(v, w) = \{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{\ell+1} \times \mathbb{T}^{n-(\ell+1)}) \mid v \le u \le w\}.$$

As in (4.1)–(4.2), for $u \in \widehat{\Gamma}_{\ell+1}$ and $i \in \mathbb{Z}$, the functions $\tau_{-i}^{\ell+1}u$ have asymptotic limits in the directions x_j, $1 \le j \le \ell$, but $J_\ell(\tau_{-i}^{\ell+1}u)$ is not yet defined. Setting $S_i^{\ell+1} = \mathbb{R}^\ell \times [i, i+1] \times \mathbb{T}^{n-(\ell+1)}$ and replacing S_0 of Chapter 4 by $S_0^{\ell+1}$ shows how J_ℓ extends to this setting and as in (4.9),

$$J_\ell(u) = c_\ell + \frac{1}{2}\|\nabla(u-v)\|^2_{L^2(S_0^{\ell+1})} + \int_{S_0{}^{\ell+1}} (F(x,u) - F(x,v))\,dx$$
$$+ \int_{\partial S_0^{\ell+1}} (u-v)\cdot\frac{\partial v}{\partial \nu} dS - \int_{S_0^{\ell+1}} (u-v)\Delta v\,dx. \tag{5.1}$$

This permits us to define $J_{\ell+1,i}(u)$ for $u \in \widehat{\Gamma}_{\ell+1}$ via

$$J_{\ell+1,i}(u) \equiv J_\ell(\tau_{-i}^{\ell+1}u) - c_\ell = J_\ell\left(u\big|_{S_i^{\ell+1}}\right) - c_\ell.$$

Continuing to follow the template of Chapter 4 yields a version of Proposition 4.10 for the current setting and the definition of $J_{\ell+1}$:

$$J_{\ell+1}(u) = \varliminf_{\substack{p\to-\infty\\ q\to\infty}} J_{\ell+1;p,q}(u).$$

An updated form of Lemma 4.14 holds, and setting

$$\Gamma_{\ell+1} \equiv \Gamma_{\ell+1}(v,w) \equiv \{u \in \widehat{\Gamma}_{\ell+1} \mid \|u-v\|_{L^2(S_i^{\ell+1})} \to 0, \text{ as}$$
$$i \to -\infty; \|u-w\|_{L^2(S_i^{\ell+1})} \to 0, i \to \infty\}$$

leads to extensions of Proposition 4.16, Lemma 4.26, Propositions 4.29 and 4.35, and Theorem 4.38. Setting

$$c_{\ell+1} = c_{\ell+1}(v_\ell, w_\ell) = \inf_{u \in \Gamma_{\ell+1}(v_\ell, w_\ell)} J_{\ell+1}(u), \tag{5.2}$$

as earlier the above results yield:

Theorem 5.3. *If F satisfies (F_1)–(F_2) and $(*)_i$ holds, $i = 0, \dots, \ell$, then:*

1^o There is a $U_{\ell+1} \in \Gamma_{\ell+1}$ such that $J_{\ell+1}(U_{\ell+1}) = c_{\ell+1}$, i.e.,

$$\mathcal{M}_{\ell+1} \equiv \mathcal{M}_{\ell+1}(v_\ell, w_\ell) \equiv \{u \in \Gamma_{\ell+1}(v_\ell, w_\ell) | J_{\ell+1}(u) = c_{\ell+1}\} \neq \emptyset.$$

2^o Any $U \in \mathcal{M}_{\ell+1}$ satisfies

(a) U is a solution of (PDE);
(b) $\|U - v_\ell\|_{C^2(\mathbb{R}^\ell \times [i,i+1] \times \mathbb{T}^{n-(\ell+1)})} \to 0, \quad i \to -\infty,$
$\|U - w_\ell\|_{C^2(\mathbb{R}^\ell \times [i,i+1] \times \mathbb{T}^{n-(\ell+1)})} \to 0, \quad i \to \infty,$
i.e., U is heteroclinic in $x_{\ell+1}$ from v_ℓ to w_ℓ;
(c) $v_\ell < U < \tau_{-1}^i U < w_\ell, \quad i = 1, \dots, \ell + 1.$

3^o $\mathcal{M}_{\ell+1}$ is an ordered set.

The remaining results of Chapter 4 also have extensions here. Thus Theorem 4.50 extends to:

Theorem 5.4. *Suppose F satisfies (F_1)–(F_2), $(*)_i$ holds, $0 \le i \le \ell - 1$, and $v, w \in \mathcal{M}_\ell$ with $v \neq w$. Then $\mathcal{M}_{\ell+1}(v, w) \neq \emptyset$ iff v and w are adjacent members of $\mathcal{M}_\ell$.*

Proof. As earlier.

The continuity result for $(*)_1$ (Proposition 4.52) and the genericity result for $(*)_1$ (Theorem 4.58) carry over to $(*)_\ell$, and as in Theorem 4.63, we have:

Theorem 5.5. *Let F satisfy (F_1)–(F_2) and let $(*)_i$ hold, $0 \le 1 \le \ell - 1$. If $u \in \mathcal{M}_\ell(v_{\ell-1}, w_{\ell-1})$, then u is minimal and WSI.*

Proof. As earlier.

Likewise, there is a version of Proposition 4.79 here. As was noted in Chapter 4, Proposition 4.79 could be viewed as a weak extension of 2^o of Theorem 3.60. However, the most natural extension of that result fails. This situation will be studied in Section 5.2.

5.2 Other Coordinate Systems

Consider $\omega = \sum_{i=1}^n p_i e_i$, where $p_i \in \mathbb{Z}$. Then $F(x + \omega, z) = F(x, z)$ for any $(x, z) \in \mathbb{R}^{n+1}$. Suppose $\omega_i = \sum_{j=1}^n \alpha_{ij} e_j$ with $\alpha_{ij} \in \mathbb{Z}$, $1 \le i, j \le n$, and the vectors ω_i are linearly independent. Using the standard Gram–Schmidt process, it can be assumed that the ω_i are orthogonal and for fixed i, the components α_{ij} of ω_i have no common factor. Now one can seek solutions of (PDE) that are periodic in the directions ω_i, i.e., $u(x + \omega_i) = u(x)$, $1 \le i \le n$. For brevity, set $\omega = (\omega_1, \dots, \omega_n)$,

$$\mathcal{R} = \mathcal{R}(\omega) = \left\{ \sum_{i=1}^m t_i \omega_i \mid 0 \le t_i \le 1, \quad 1 \le i \le n \right\}$$

and set

$$\Gamma_0(\omega) = \{u \in W^{1,2}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}) \mid u(x + \omega_i) = u(x),\ \ 1 \le i \le n\}. \tag{5.6}$$

For $u \in \Gamma_0(\omega)$, let

$$J_0^\omega(u) = \int_{\mathcal{R}} L(u)dx \tag{5.7}$$

and set

$$c_0(\omega) = \inf_{u \in \Gamma_0(\omega)} J_0^\omega(u). \tag{5.8}$$

As in Theorem 1.6, there is a set $\mathcal{M}_0(\omega)$ of minimizers of this variational problem and $\mathcal{M}_0(\omega)$ is ordered. Moreover, continuing as in Chapters 2–4 and part (A) of this chapter produces versions of our earlier results with $e_1, \ldots, e_n$ replaced by ω. However, as Proposition 2.2 hints, this generalization of the previous results is not as extensive as it first appears. In particular:

Lemma 5.9. $\mathcal{M}_0(\omega) = \mathcal{M}_0(e_1, \ldots, e_n)$.

Proof. Let $u \in \mathcal{M}_0(\omega)$. Thus for each i, $u(x + e_i) \in \Gamma_0(\omega)$ and

$$J_0^\omega(u(x + e_i)) = \int_{\mathcal{R}+\{e_i\}} L(u)dx = J_0^\omega(u) = c_0(\omega),$$

so $u \in \mathcal{M}_0(\omega)$. Since $\mathcal{M}_0(\omega)$ is ordered, we have (a) $u(x + e_i) > u(x)$, (b) $u(x + e_i) < u(x)$, or (c) $u(x + e_i) = u(x)$. Suppose (a) holds. Since $e_i = \sum_k p_{ik}\omega_k$ for some $p_{ik} \in \mathbb{Q}$, there is a $j \in \mathbb{N}$ such that $jp_{ik} \in \mathbb{Z}$, $1 \le k \le n$. Now (a) implies

$$u(x) < u(x + e_i) < \cdots < u(x + je_i) = u(x),$$

a contradiction. Similarly, (b) cannot occur. Thus (c) holds for $1 \le i \le n$, so $u \in \Gamma_0(e_1, \ldots, e_n)$. Moreover, $u \in \mathcal{M}_0(\omega)$ implies that u is minimal. Therefore as in the proof of 2^o of Theorem 3.60, $u \in \mathcal{M}_0(e_1, \ldots, e_n)$.

Conversely, $u \in \mathcal{M}_0(e_1, \ldots, e_n)$ implies

$$J_0^\omega(u) = (\text{vol}\,\mathcal{R})c_0 = \det(\alpha_{ij})c_0 = c_0(\omega),$$

so $u \in \mathcal{M}_0(\omega)$.

With Lemma 5.9 in hand, when $(*)_0$ holds, following the arguments of Chapters 2–3 yields a class of functions $\Gamma_1(v_0, w_0; \omega)$ with ω_i replacing e_i, etc. Likewise, there are a corresponding renormalized functional $J_1^\omega(u)$ and minimization value

$$c_1(v_0, w_0; \omega) = \inf_{u \in \Gamma_1(v_0, w_0; \omega)} J_1^\omega(u.) \tag{5.10}$$

This leads to a version of Theorem 3.2 for the current setting and shows that

$$\mathcal{M}_1(v_0, w_0; \omega) = \{u \in \Gamma_1(v_0, w_0; \omega) | J_1^\omega(u) = c_1(v_0, w_0; \omega)\} \neq \emptyset$$

and that it is an ordered set of solutions of (PDE). However, as the next proposition shows, the flexibility with respect to ω again is less than it first appears to be.

Proposition 5.11. *Let $\omega = (\omega_1, \dots, \omega_n)$ and $\hat{\omega} = (\hat{\omega}_1, \dots, \hat{\omega}_n)$ be admissible sets of orthogonal vectors. Then*

$$\mathcal{M}_1(v_0, w_0; \omega) = \mathcal{M}_1(v_0, w_0; \hat{\omega}) \iff \omega_1 = \hat{\omega}_1,$$

i.e., $\mathcal{M}_1(v_0, w_0; \cdot)$ is determined by ω_1.

Proof. Suppose $u \in \mathcal{M}_1(v_0, w_0; \omega) = \mathcal{M}_1(v_0, w_0; \hat{\omega})$. Since $\hat{\omega}_2 = \sum_{k=1}^n q_{2k}\omega_k$, $j \in \mathbb{N}$ can be chosen so that $jq_{2k} \in \mathbb{Z}$, $1 \le k \le n$. Then for $\ell \in \mathbb{N}$,

$$u(x) = u(x + \hat{\omega}_2) = u(x + j\hat{\omega}_2) = u(x + jq_{21}\omega_1) = u(x + \ell jq_{21}\omega_1), \quad (5.12)$$

so

$$v_0(x) < u(x + jq_{21}\omega_1) = u(x + \ell jq_{21}\omega_1) < w_0(x). \quad (5.13)$$

If $q_{21} \neq 0, u(x + \ell jq_{21}\omega_1) \to v_0(x)$ or $w_0(x)$ as $\ell \to \infty$, contrary to (5.13). Therefore q_{21} and similarly $q_{i1} = 0$, $2 \le i \le n$. Thus ω_1 lies in the orthogonal complement of span $(\hat{\omega}_2, \dots, \hat{\omega}_n)$, i.e., $\omega_1 = \gamma\hat{\omega}_1$ for some $\gamma \in \mathbb{R}$. But $\omega_1 = \sum_1^n a_i e_i$ and $\hat{\omega}_1 = \sum_1^n \hat{a}_i e_i$. Hence $a_i = \gamma\hat{a}_i$. The earlier normalization that the components of $\hat{\omega}_i$ have no common factors implies $\gamma = \pm 1$. If $\gamma = -1$ and $\xi \in \{\omega_1, \hat{\omega}_1\}$,

$$u(x + \ell\xi) \to w_0(x) \quad \text{as} \quad \ell \to \infty, \quad (5.14)$$

but

$$u(x + \ell\hat{\omega}_1) = u(x - \ell\omega_1) \to v_0(x) \quad (5.15)$$

as $\ell \to \infty$, contrary to (5.14). Thus $\gamma = 1$ and $\omega_1 = \hat{\omega}_1$.

Next suppose $\omega_1 = \hat{\omega}_1$ and let $u \in \mathcal{M}_1(v_0, w_0; \omega)$. Then $u(x + \hat{\omega}_2) \in \Gamma_1(v_0, w_0; \omega)$, and since $\hat{\omega}_2 \in$ span $(\omega_2, \dots, \omega_n)$,

$$J_1^{\omega}(u)((x + \hat{\omega}_2)) = J_1^{\omega}(u),$$

so $u(x + \hat{\omega}_2) \in \mathcal{M}_1(v_0, w_0; \omega)$, an ordered set. Therefore (a) $u(x + \hat{\omega}_2) > u(x)$, (b) $u(x + \hat{\omega}_2) < u(x)$, or (c) $u(x + \hat{\omega}_2) = u(x)$. If (a) occurs, as in (5.12), for appropriate j,

$$u(x) < u(x + j\hat{\omega}_2) = u(x). \quad (5.16)$$

Thus (a) and likewise (b) cannot occur. A similar argument shows that $u(x + \hat{\omega}_i) = u(x)$, $2 \le i \le n$. Consequently $u \in \Gamma_1(v_0, w_0; \hat{\omega})$. Moreover, by the analogue here of Theorem 3.60 1^o, u is minimal and WSI. Then 2^o of that theorem implies $u \in \mathcal{M}_1(v_0, w_0; \hat{\omega})$. Reversing the roles of ω and $\hat{\omega}$ then yields $\mathcal{M}_1(v_0, w_0; \omega) = \mathcal{M}_1(v_0, w_0; \hat{\omega})$.

Proposition 5.11 shows that the sets $\mathcal{M}_1(v_0, w_0; \omega)$ are more properly denoted by $\mathcal{M}_1(v_0, w_0; \omega_1)$. In particular, to get heteroclinics at the next level of complexity merely requires a gap in $\mathcal{M}_1(v_0, w_0; \omega_1)$ independently of the choice of

$\omega_2, \ldots, \omega_n$. Thus condition $(*)_1$ depends only on ω_1 and will be denoted by $(*)_1(\omega_1)$. If it holds, denoting the associated gap pair by $v_1(\omega_1)$, $w_1(\omega_1)$ and defining $\Gamma_2(v_1(\omega_1), w_1(\omega_1); \omega)$, J_2^ω, $c_2(\omega)$ in the natural fashion leads to a version of Theorem 4.40 for this setting. A priori, the set $\mathcal{M}_2(v_1(\omega_1), w_1(\omega_1); \omega)$ depends on $\omega_2, \ldots, \omega_n$, but again as for Proposition 5.11, it depends only on ω_2:

Proposition 5.17. *Let* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ *and* $\hat{\omega} = (\omega_1, \hat{\omega}_2, \ldots, \hat{\omega}_n)$. *Then*

$$\mathcal{M}_2(v_1(\omega_1), w(\omega_1); \omega) = \mathcal{M}_2(v_1(\omega_1), w_1(\omega_1); \hat{\omega}) \text{ iff } \omega_2 = \hat{\omega}_2.$$

Proof. That the equality of the sets implies $\omega_2 = \hat{\omega}_2$ follows as in the proof of Proposition 5.11 with small modifications. For the converse, suppose $\omega_2 = \hat{\omega}_2$ and $u \in \mathcal{M}_2(v_1(\omega_1), w_1(\omega_1); \omega)$. Then again as earlier, $u(x + \hat{\omega}_i) = u(x), 3 \le i \le n$, so $u \in \widehat{\Gamma}_2(v_1(\omega_1), w_1(\omega_1))$. We claim that $u \in \Gamma_2(v_1(\omega_1), w_1(\omega_1); \hat{\omega})$. To show this requires proving

$$\|u - v_1(\omega_1)\|_{L^2(S_i^{\hat{\omega}})} \to 0, \quad i \to -\infty, \tag{5.18}$$

$$\|u - w_1(\omega_1)\|_{L^2(S_i^{\hat{\omega}})} \to 0, \quad i \to \infty. \tag{5.19}$$

Here $S_i^{\hat{\omega}}$ is the analogue of the earlier strips S_i. Thus $S_i^{\hat{\omega}} = S_0^{\hat{\omega}} + i\omega_2$ and

$$S_0^{\hat{\omega}} = \left\{ t_1\omega_1 + t_2\omega_2 + \sum_3^n t_i\hat{\omega}_i \mid t_1 \in \mathbb{R}, \quad 0 \le t_i \le 1, 2 \le i \le n \right\}.$$

Note that

$$\begin{aligned} S_0^{\hat{\omega}} &= \left\{ t_1\omega_1 + t_2\omega_2 + \sum_3^n t_i q_{ik}\omega_k \mid t_1 \in \mathbb{R}, \quad 0 \le t_i \le 1, 2 \le i \le n \right\} \\ &\subset \{t_i\omega_i | t_1 \in \mathbb{R}, 0 \le t_2 \le 1, |t_i| \le j\} \equiv S^* \end{aligned}$$

for some $j \in \mathbb{N}$. Therefore

$$\|u - v_1(\omega_1)\|_{L^2(S_i^{\hat{\omega}})} \le \|u - v_1(\omega_1)\|_{L^2(S^* + i\omega_2)}, \tag{5.20}$$

and since $u \in \Gamma_2(v_1(\omega_1), w_1(\omega_1), \omega)$, the right-hand side of (5.20) goes to 0 as $i \to -\infty$. Thus (5.18) and similarly (5.19) are satisfied. Consequently, $u \in \Gamma_2(v_1(\omega_1), w_1(\omega_1), \hat{\omega})$. Since u is also minimal and WSI, by a variant of Proposition 4.79, $u \in \mathcal{M}_2(v_1(\omega_1), w_1(\omega_1), \omega)$, and Proposition 5.17 is proved.

Continuing in this fashion leads to further solutions of (PDE) as in (A) of this chapter with properties as in Chapters 2–4 as well as corresponding versions of Proposition 5.17.

5.3 Generalizations to $\alpha \in \mathbb{Q}^n$

So far, only the case of the rotation vector $\alpha = 0$ has been treated. This section indicates how our earlier results extend to $\alpha \in \mathbb{Q}^n$.

Let $r \in \mathbb{N}^n$ and $s \in \mathbb{Z}^n$. Suppose u^* satisfies

$$u^*(x + r_i e_i) = u^*(x) + s_i, \quad 1 \le i \le n. \tag{5.21}$$

By Theorem 1.1, if such a u^* is a solution of (PDE) that is minimal and WSI, there are an $\alpha \in \mathbb{Q}^n$ and $M > 0$ such that

$$|u^*(x) - \alpha \cdot x| \le M$$

for all $x \in \mathbb{R}^n$. By (5.21) for $1 \le i \le n$ and $k \in \mathbb{Z}$,

$$|u^*(x + kr_i e_i) - \alpha \cdot (x + kr_i e_i)| = |u^*(x) + ks_i - \alpha \cdot x - \alpha_i k r_i|, \tag{5.22}$$

and (5.22) is bounded in k iff

$$\alpha_i = s_i / r_i, \quad 1 \le i \le n. \tag{5.23}$$

Thus given $\alpha \in \mathbb{Q}^n$, choosing $r \in \mathbb{N}^n$, $s \in \mathbb{Z}^n$ with r_i, s_i relatively prime and satisfying (5.23), solutions of (PDE) having rotation vector α can be sought in the class of functions satisfying (5.21). For u^* in this class, set $u = u^* - \alpha \cdot x$. Then for $1 \le i \le n$,

$$u(x + r_i e_i) = u^*(x + r_i e_i) - \alpha \cdot (x + r_i e_i) = u^*(x) + s_i - \alpha \cdot x - \alpha_i r_i = u(x), \tag{5.24}$$

i.e., u is r_i-periodic in x_i, or in the notation of Chapter 2, $u \in \Gamma_0(r)$, where $r = (r_1, \ldots, r_n)$. Moreover, if u^* satisfies (PDE),

$$-\Delta u + F_u(x, u + \alpha \cdot x) = 0. \tag{5.25}$$

Thus to find solutions u^* of (PDE) of rotation vector α, it suffices to find $u \in \Gamma_0(r)$ satisfying (5.25).

With these observations, results paralleling our main earlier theorems obtain for each $\alpha \in \mathbb{Q}^n$. We will indicate them for the simplest cases and make some remarks about the more general ones. To begin using suggestive notation, the results of Chapters 1–3 become the following: There is an ordered set of solutions of (PDE), $\mathcal{M}_0^\alpha$, satisfying (5.25). In addition, whenever there is a gap pair $v_\alpha < w_\alpha$ in $\mathcal{M}_0^\alpha$, there is a solution of (PDE), U_1^α, lying in the gap, heteroclinic in x_1 from v_α to w_α and satisfying (5.24) for $i = 2, \ldots, n$. The function U_1^α is a minimizer of an associated functional J_1^α defined on $\Gamma_1^\alpha(v_\alpha, w_\alpha)$, and the set of such minimizers $\mathcal{M}_1^\alpha(v_\alpha, w_\alpha)$ is nonempty. Conversely if $\mathcal{M}_1^\alpha(v_\alpha, w_\alpha) \neq \emptyset$, v_α, w_α are adjacent members of $\mathcal{M}_0^\alpha$. Moreover, u is a solution of (PDE) having rotation vector α satisfying (5.23) for $2 \le i \le n$ which is minimal and WSI iff $u \in \mathcal{M}_0^\alpha$ or $u \in \mathcal{M}_1^\alpha(v_\alpha, w_\alpha) \cup \mathcal{M}_1^\alpha(w_\alpha, v_\alpha)$ for some adjacent pair v_α, w_α in $\mathcal{M}_0^\alpha$. The changes required of the material in Chapters 1–3 to obtain these results are minor. Therefore the new classes of functions and functionals that are needed will be defined, but most proofs will be omitted.

To introduce $\mathcal{M}_0^\alpha$, a version of Proposition 2.2 will be needed, so for the moment we work with $r \in \mathbb{N}^n$ and $s \in \mathbb{Z}^n$ rather that α. Define

$$\Gamma_0^{r,s} = \{u^* \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \mid u^* \text{ satisfies (5.21)}\}.$$

Setting $\alpha_i = s_i/r_i$, $1 \le i \le n$, by (5.24),

$$\Gamma_0^{r,s} = \{u + \alpha \cdot x \mid u \in \Gamma_0(r)\} = \Gamma_0(r) + \alpha \cdot x.$$

For $u \in \Gamma_0(r)$ and J_0^r as in Chapter 2, define

$$c_0^{r,s} = \inf_{u \in \Gamma_0^r} J_0^r(u + \alpha \cdot x). \tag{5.26}$$

Set

$$\mathcal{M}_0^{r,s} = \{u + \alpha x \mid u \in \Gamma_0^r \text{ and } J_0^r(u + \alpha x) = c_0^{r,s}\}.$$

In [1], Moser proved

Theorem 5.27. *1°* $\mathcal{M}_0^{r,s} \neq \emptyset$.
2° Any $u^* = u + \alpha \cdot x \in \mathcal{M}_0^{r,s}$ *is a solution of (PDE) that is minimal and WSI.*
3° $\mathcal{M}_0^{r,s}$ *is an ordered set.*
4° For $k \in \mathbb{N}^n$ *and* $t \in \mathbb{Z}^n$, *set* $\hat{k}(t) = (k_1 t_1, \ldots, k_n t_n)$. *Then* $\mathcal{M}_0^{\hat{k}(r),\hat{k}(s)} = \mathcal{M}_0^{r,s}$ *and*

$$c_0^{\hat{k}(r),\hat{k}(s)} = \left(\prod_1^n k_i\right) c_0^{r,s}.$$

Proof. 1^o–3^o are proved as earlier. For 4^o, let $u + \alpha x \in \Gamma_0^{\hat{k}(r),\hat{k}(s)}$. Then using (F_2), a computation shows that for $1 \le i \le n$, $\tau_{r_i}^i u + \alpha \cdot x \in \Gamma_0^{\hat{k}(r),\hat{k}(s)}$. Therefore $u + \alpha \cdot x \in \mathcal{M}_0^{\hat{k}(r),\hat{k}(s)}$ implies $\tau_{r_i}^i u + \alpha \cdot x \in \mathcal{M}_0^{\hat{k}(r),\hat{k}(s)}$. Note that 3^o is equivalent to the statement that

$$\{u \in \Gamma_0(r) \mid u + \alpha \cdot x \in \mathcal{M}_0^{r,s}\} \equiv \mathcal{M}_0(r)$$

is ordered. Hence, (i) $\tau_{r_i}^i u = u$, (ii) $\tau_{r_i}^i u > u$, or (iii) $\tau_{r_i}^i u < u$. Possibilities (ii) and (iii) are excluded as in Proposition 2.2, so (i) holds. Thus $u \in \Gamma_0(r)$ and it satisfies (5.21) with r, s.

To continue, henceforth for a given $\alpha \in \mathbb{Q}^n$, choose the unique $r \in \mathbb{N}^n$ and $s \in \mathbb{Z}^n$ such that $\alpha_i = s_i/r_i$ and s_i, r_i are relatively prime, $1 \le i \le n$. Further set $\Gamma_0^\alpha \equiv \Gamma_0^{r,s}$, $c_0^\alpha \equiv c_0^{r,s}$, and $\mathcal{M}_0^\alpha \equiv \mathcal{M}_0^{r,s}$.

Assume that

$$\text{there are adjacent } v_0^\alpha, w_0^\alpha \in \mathcal{M}_0(r) \text{ with } v_0^\alpha < w_0^\alpha. \tag{$(*)_0^\alpha$}$$

We seek a solution $U^* = U + \alpha \cdot x$ of (PDE) with U heteroclinic in x_1 from v_0^α to w_0^α. To formulate a variational problem for U, replace $\mathbb{T}^{n-1}$ and T_i of Chapter 2 by $\mathbb{R}/[0, r_2] \times \cdots \times \mathbb{R}/[0, r_n] \equiv \mathbb{T}_\alpha^{n-1}$ and $[ir_1, (i+1)r_1] \times \mathbb{T}_\alpha^{n-1} \equiv \mathbb{T}_i^\alpha$. Then for $v, w \in \mathcal{M}_0(r)$, define

$$\widehat{\Gamma}_1^\alpha \equiv \widehat{\Gamma}_1^\alpha(v, w) \equiv \left\{u \in W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{T}_\alpha^{n-1}) \mid u \text{ lies between } v \text{ and } w\right\}.$$

For $u \in \widehat{\Gamma}_1^\alpha$ and $i \in \mathbb{Z}$, set

$$J_{1,i}^\alpha(u) = \int_{\mathbb{T}_i^\alpha} L(u + \alpha \cdot x)dx - c_0^\alpha$$

and

$$J_{1;p,q}^\alpha(u) = \sum_p^q J_{1,i}^\alpha(u).$$

Then with the aid of 4^o of Theorem 5.27, the proof of Proposition 2.8 carries over to the current setting with minor modifications. This allows us to define

$$J_1^\alpha(u) = \varliminf_{\substack{p \to -\infty \\ q \to \infty}} J_{1;p,q}^\alpha(u),$$

and Lemma 2.22 extends to this functional. Next defining

$$\begin{aligned} \Gamma_1^\alpha \equiv \Gamma_1^\alpha(v, w) = &\left\{u \in \widehat{\Gamma}_1^\alpha \mid \|u - v\|_{L^2(\mathbb{T}_i^\alpha)} \to 0, i \to -\infty,\right. \\ &\left.\|u - w\|_{L^2(\mathbb{T}_i^\alpha)} \to 0, i \to \infty\right\}, \end{aligned}$$

Proposition 2.24 extends to J_1^α and Γ_1^α as do the remaining results of Chapter 2 with small changes. For example, in Corollary 2.49, $u \le \tau_{-1}^1 u$ is replaced by $u \le \tau_{-r_1}^1 u$, the space $\Gamma_1(v)$ becomes

$$\begin{aligned} &\Gamma_1^\alpha(v) \equiv \left\{u \in \widehat{\Gamma}_1^\alpha(v - 1, v + 1) \mid \|u - v\|_{L^2(\mathbb{T}_i^\alpha)} \to 0, \text{ as } |i| \to \infty\right\}, \\ &c_1^\alpha(u) \equiv \inf_{u \in \Gamma_1^\alpha(u)} J_1^\alpha(u), \end{aligned}$$

and

$$\mathcal{M}_1^\alpha(v) = \{u \in \Gamma_1^\alpha(v) \mid J_1^\alpha(u) = c_1^\alpha(v)\}.$$

These preliminaries lead to:

Theorem 5.28. *If F satisfies (F_1)–(F_2) and $(*)_0^\alpha$ holds:*

1º There is a $U_1^\alpha \in \Gamma_1^\alpha(v_0^\alpha, w_0^\alpha)$ such that $J_1^\alpha(U_1^\alpha) = c_1^\alpha$, i.e., $\mathcal{M}_1^\alpha = \mathcal{M}_1^\alpha(v_0^\alpha, w_0^\alpha) \equiv \{u \in \Gamma_1^\alpha(v_0^\alpha, w_0^\alpha) \mid J_1^\alpha(u) = c_1^\alpha\} \neq \emptyset$.
2º If $U \in \mathcal{M}_1^\alpha$,

- *(a) $U + \alpha \cdot x$ is a solution of (PDE),*
- *(b) $\|U - v_0^\alpha\|_{C^2(\mathbb{T}_i^\alpha)} \to 0$, $i \to -\infty$,*
  $\|U - w_0^\alpha\|_{C^2(\mathbb{T}_i^\alpha)} \to 0$, $i \to \infty$,
- *(c) $v_0^\alpha < U < \tau_{-r_1}^1 U < w_0^\alpha$.*

3º $\mathcal{M}_1^\alpha$ is an ordered set.

The proof of Theorem 5.28 follows that of Theorem 3.2.

With the above observations, it is straightforward to extend the remaining results of Chapters 3–4 to $\alpha \in \mathbb{Q}^n$. Likewise, Sections 5.1 and 5.2 of this chapter carry over to $\alpha \in \mathbb{Q}^n$. For example, for Section 5.2, we replace $(e_1, \ldots, e_n)$ by $(\omega_1, \ldots, \omega_n)$ in (5.16).

Part II
Shadowing Results

Chapter 6
The Simplest Cases

In the second part of this memoir, the existence and variational characterizations of the basic solutions of (PDE) that were found in Part I will be used to construct more complex solutions. The new solutions are near formal concatenations of the basic solutions. Hence in the terminology of dynamical systems, they shadow basic solutions, while in the language that has been used in other related settings, they are "multibump" solutions of (PDE). The term "multitransition" solution is more accurate, and it will be used here.

Two of the simplest cases will be studied first. To describe them, observe that by $(*)_0$, $\mathcal{M}_1(v_0, w_0) \neq \emptyset$ and therefore

$$\mathcal{M}_1(v_0 + 1, w_0 + 1) = \{1 + u \mid u \in \mathcal{M}_1(v_0, w_0)\} \neq \emptyset.$$

This suggests trying to find solutions of (PDE) that are heteroclinic in x_1 from v_0 to $w_0 + 1$, are 1-periodic in $x_2, \ldots, x_n$, and shadow members of $\mathcal{M}_1(v_0, w_0)$ and $\mathcal{M}_1(v_0 + 1, w_0 + 1)$. It will be shown that there are infinitely many such solutions of (PDE), provided that $(*)_1$ holds, i.e., $\mathcal{M}_1(v_0, w_0)$ has gaps. These heteroclinics u, whose existence was alluded to in Remark 3.41, are strictly 1-monotone in x_1 and also possess some minimality properties, but they are not minimal as in Chapter 1.

Next observe that $(*)_0$ implies that both $\mathcal{M}_1(v_0, w_0)$ and $\mathcal{M}_1(w_0, v_0)$ are nonempty. Thus one can seek solutions of (PDE) homoclinic to v_0 (or to w_0) that shadow members of $\mathcal{M}_1(v_0, w_0)$ and $\mathcal{M}_1(w_0, v_0)$. Under the further assumptions that $\mathcal{M}_1(v_0, w_0)$ and $\mathcal{M}_1(w_0, v_0)$ have gaps, it will be shown that there are infinitely many such solutions. Unlike the previous case, they are not monotone but again possess local minimality properties. Consequently, here we leave the realm of solutions that are minimal and WSI.

As in Part I, the main tools for obtaining these new heteroclinic and homoclinic solutions are minimization and comparison arguments. However, in contrast to the earlier settings, the new variational problems involve additional integral constraints that force admissible functions to have the shadowing properties we seek. Such constrained variational approaches have been used in dynamical systems settings by Mather [6] and others and also for partial differential equations as in [7, 8].

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DOI 10.1007/978-0-8176-8117-3_6,

There are different kinds of shadowing results one can attempt to find. The most precise sort of result, which requires the greatest technical effort, is to seek solutions that are globally near a given pair of isolated basic solutions (or if the basic solutions are not isolated, the new solutions should shadow the corresponding respective components of solutions). A less onerous approach gives shadowing orbits in a "controlled region" of the function space under consideration, a region that may contain many basic solutions. By a controlled region, we mean that constraints are imposed on the functions that require them to have the form we seek. Our results are mainly of this latter type.

Turning to the two cases that are the current focus, the second is simpler in that it concerns only solutions lying in the gap between v_0 and w_0. The first case deals with the region between v_0 and $w_0 + 1$, which may contain a complicated set of periodic, heteroclinic, or homoclinic solutions of (PDE). The simpler case will be treated in Chapters 6–8 and the monotone case in Chapter 9. To formulate the main result for two-transition heteroclinic solutions of (PDE) between v_0 and w_0, assume $(*)_0$ and also $(*)_1$ for $\mathcal{M}_1(v_0, w_0)$ and $\mathcal{M}_1(w_0, v_0)$. Define

$$\begin{cases} \rho_-(u) = \|u - v_0\|_{L^2(T_0)}, \\ \rho_+(u) = \|u - w_0\|_{L^2(T_0)}. \end{cases} \tag{6.1}$$

By Theorem 3.2, $\mathcal{M}_1(v_0, w_0)$ and $\mathcal{M}_1(w_0, v_0)$ are ordered sets. Therefore ρ_- is strictly increasing on $\mathcal{M}_1(v_0, w_0)$, and $\mathcal{M}_1(w_0, v_0)$ while ρ_+ is strictly decreasing on these two sets. Set $\overline{\rho} = \|w_0 - v_0\|_{L^2(T_0)}$. Choose constants $\rho_i \in (0, \overline{\rho})$, $1 \le i \le 4$, such that

$$\begin{cases} \rho_1 \notin \rho_-(\mathcal{M}_1(v_0, w_0)), & \rho_2 \notin \rho_+(\mathcal{M}_1(v_0, w_0)), \\ \rho_3 \notin \rho_+(\mathcal{M}_1(w_0, v_0)), & \rho_4 \notin \rho_-(\mathcal{M}_1(w_0, v_0)). \end{cases} \tag{6.2}$$

Let $\ell \in \mathbb{N}$ and $m \in \mathbb{Z}^4$ with

$$m_1 < m_2 < m_2 + 2\ell < m_3 < m_4. \tag{6.3}$$

Now the class of admissible functions for our first minimization problem can be introduced. Set

$$Y_{m,\ell} \equiv Y_{m,\ell}(v_0, w_0) \equiv \{u \in \widehat{\Gamma}_1(v_0, w_0) \mid u \text{ satisfies (6.5)–(6.6)}\}, \tag{6.4}$$

where

$$\begin{aligned} &\text{(i)} \;\; \rho_-(\tau_{-i}^1 u) \le \rho_1, && m_1 - \ell \le i \le m_1 - 1, \\ &\text{(ii)} \;\; \rho_+(\tau_{-i}^1 u) \le \rho_2, && m_2 \le i \le m_2 + \ell - 1, \\ &\text{(iii)} \;\; \rho_+(\tau_{-i}^1 u) \le \rho_3, && m_3 - \ell \le i \le m_3 - 1, \\ &\text{(iv)} \;\; \rho_-(\tau_{-i}^1 u) \le \rho_4, && m_4 \le i \le m_4 + \ell - 1, \end{aligned} \tag{6.5}$$

and

$$\|\tau_{-i}^{1} u - v_0\|_{L^2(T_i)} \to 0, \quad |i| \to \infty. \tag{6.6}$$

Define

$$b_{m,\ell} \equiv b_{m,\ell}(v_0, w_0) \equiv \inf_{u \in Y_{m,\ell}} J_1(u). \tag{6.7}$$

The main result of this section is:

Theorem 6.8. *Let F satisfy (F_1)–(F_2). Assume that $(*)_0$ holds and also $(*)_1$ for $\mathcal{M}_1(v_0, w_0)$ and $\mathcal{M}_1(w_0, v_0)$. Then for each sufficiently large $\ell \in \mathbb{N}$, there is a $U = U_{m,\ell} \in Y_{m,\ell}$ such that $J_1(U) = b_{m,\ell}$. If in addition $m_2 - m_1$ and $m_4 - m_3$ are sufficiently large, U is a solution of (PDE) and*

$$\|U - v_0\|_{C^2(T_i)} \to 0 \quad as\ |i| \to \infty. \tag{6.9}$$

Remark 6.10. (i) Applying Theorem 6.8 with larger and larger choices for ℓ, $m_2 - m_1$ and $m_4 - m_3$ produces infinitely many distinct solutions of (PDE).
(ii) Stronger statements about shadowing can be made. For example, U will be close to v_0 in $\|\cdot\|_{W^{1,2}(T_i)}$ for $i \le m_1$ and $i \ge m_4$ and to w_0 in $\|\cdot\|_{W^{1,2}(T_i)}$ for $m_2 \le i \le m_3$.

The proof of Theorem 6.8 will be given in Chapter 7. It requires a few preliminaries, which will be stated and proved in this section. Using Theorem 6.8 as the main tool, the existence of multitransition solutions will be studied in Chapter 8.

Lemma 6.11. $c_1(v_0, w_0) + c_1(w_0, v_0) > 0$.

Proof. Let $V \in \mathcal{M}_1(v_0, w_0)$ and $W \in \mathcal{M}_1(w_0, v_0)$. Set $\Phi = \max(V, W)$ and $\Psi = \min(V, W)$. Then $\Phi \in \Gamma_1(w_0) \setminus \{w_0\}$ and $\Psi \in \Gamma_1(v_0) \setminus \{v_0\}$. Therefore by Theorem 2.72, $J_1(\Phi)$, $J_1(\Psi) > 0$, and as in (2.79)–(2.80),

$$0 < J_1(\Phi) + J_1(\Psi) = J_1(V) + J_1(W) = c_1(v_0, w_0) + c_1(w_0, v_0). \tag{6.12}$$

The next result is related to Lemma 6.11 and provides an estimate useful in future comparison arguments. Set

$$X_0 \equiv \bigcup_{i=-2}^{2} T_i.$$

Proposition 6.13. *Suppose $(*)_0$ holds. Let $\gamma > 0$. Then for any $u \in \Gamma_1(v_0)$ (resp. $u \in \Gamma_1(w_0)$) satisfying*

$$\|u - v_0\|_{L^2(X_0)} \ge \gamma \quad (resp. \|u - w_0\|_{W^{1,2}(X_0)} \ge \gamma), \tag{6.14}$$

there is a $\beta = \beta(\gamma) > 0$ independent of u such that $J_1(u) \ge \beta$.

Proof. The proofs are the same for the v_0 and w_0 cases, so the former case will be proved. Set

$$\mathcal{Y} = \{u \in \Gamma_1(v_0) \mid u \text{ satisfies (6.14)}\}$$

and

$$c(\mathcal{Y}) = \inf_{u \in \mathcal{Y}} J_1(u). \tag{6.15}$$

Then by Theorem 2.72,

$$0 = c_1(v_0) \le c(\mathcal{Y}) < \infty. \tag{6.16}$$

If $c(\mathcal{Y}) > 0$, Proposition 6.13 follows with $\beta(\gamma) = c(\mathcal{Y})$. Hence it suffices to show that $c(\mathcal{Y}) = 0$ is not possible. Thus suppose $c(\mathcal{Y}) = 0$ and let (u_k) be a minimizing sequence for (6.15). Then (u_k) is also a minimizing sequence for (2.71). Since $\mathcal{Y} \subset \Gamma_1(v_0)$, which satisfies (Y_1^1)–(Y_2^1), by Propositions 2.50 and 2.64, it can be assumed that there is a $P \in W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ such that $u_k \to P$ in $W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ as $k \to \infty$, thereby satisfying (6.14), and P is a solution of (PDE). Moreover, as, e.g., in (3.6)–(3.7), $J_1(P) < \infty$. Consider $\Phi_k = \max(u_k, \tau_{-1}^1 u_k)$ and $\Psi_k = \min(u_k, \tau_{-1}^1 u_k)$. Then as in (3.14) (with $c_1 = 0$) and the argument following it, Φ_k, Ψ_k converge to $\Phi = \max(P, \tau_{-1}^1 P)$ and $\Psi = \min(P, \tau_{-1}^1 P)$, which are solutions of (PDE) with $\Phi \ge \Psi$ and either (i) $\Phi \equiv \Psi$ or (ii) $\Phi > \Psi$ on $\mathbb{R} \times \mathbb{T}^{n-1}$. If (i) is satisfied, $P = \tau_{-1}^1 P$, so $P \in \Gamma_0$. Therefore $J_1(P) < \infty$ implies $P \in \{v_0, w_0\}$. By (6.14), $P = v_0$ is not possible. Thus (i) implies $P = w_0$. If (ii) is valid, (a) $\tau_{-1}^1 P > P$ or (b) $P > \tau_{-1}^1 P$. Alternative (a) shows that $P \in \Gamma_1(v_0, w_0)$, while (b) implies $P \in \Gamma_1(w_0, v_0)$. A similar argument applies in either event, so suppose (a) is satisfied. Then by Proposition 2.24,

$$\|P - w_0\|_{W^{1,2}(T_i)} \to 0, \quad i \to \infty. \tag{6.17}$$

Note that (6.17) also is valid for case (i). Thus to verify that $c(\mathcal{Y}) > 0$ and complete the proof, it suffices to prove that (6.17) is impossible. A comparison argument exploiting Lemma 6.11 will be employed to do so.

Let $\varepsilon > 0$. Since $u_k \to P$ in $W^{1,2}(T_s)$ for each $s \in \mathbb{Z}$, (6.17) shows that there is a $q = q(\varepsilon) \in \mathbb{N}$ such that for all large $k \in \mathbb{N}$,

$$\|u_k - w_0\|_{W^{1,2}(T_q)} \le \varepsilon. \tag{6.18}$$

Define

$$f_k = \begin{cases} u_k, & x_1 \le q - 1, \\ w_0, & q \le x_1 \le q + 1, \\ u_k, & q + 2 \le x_1, \end{cases} \tag{6.19}$$

and interpolate in between as usual. Then as in (3.23), there is a function $\mu(s)$ satisfying $\mu(s) \to 0$ as $s \to 0$ and such that

$$|J_1(u_k) - J_1(f_k)| \le \mu(\varepsilon) \tag{6.20}$$

for large k. Further, choose ε so that

$$\mu(\varepsilon) < \frac{1}{2}(c_1(v_0, w_0) + c_1(w_0, v_0)). \tag{6.21}$$

Hence by (6.20)–(6.21), for large k,

$$J_1(f_k) \leq J_1(u_k) + \frac{1}{2}(c_1(v_0, w_0) + c_1(w_0, v_0)). \tag{6.22}$$

Define

$$g_k = \begin{cases} f_k, & x_1 \leq q, \\ w_0, & q \leq x_1, \end{cases} \tag{6.23}$$

and

$$h_k = \begin{cases} w_0, & x_1 \leq q, \\ f_k, & q \leq x_1. \end{cases} \tag{6.24}$$

Then

$$J_1(f_k) = J_1(g_k) + J_1(h_k) \tag{6.25}$$

and $g_k \in \Gamma_1(v_0, w_0)$, $h_k \in \Gamma_1(w_0, v_0)$. Consequently, by (6.22)–(6.25),

$$c_1(v_0, w_0) + c_1(w_0, v_0) \leq J_1(u_k) + \frac{1}{2}(c_1(v_0, w_0) + c_1(w_0, v_0)),$$

or via Lemma 6.11,

$$0 < \frac{1}{2}(c_1(v_0, w_0) + c_1(w_0, v_0)) \leq J_1(u_k) \tag{6.26}$$

for all large k. But $J_1(u_k) \to 0$ as $k \to \infty$, contrary to (6.26). Thus $c(\mathcal{Y}) > 0$, and Proposition 6.13 is proved.

The next result provides a crucial tool for future cutting and pasting arguments and for analyzing asymptotic behavior. Define $X_i = \bigcup_{j=-2}^{2} T_{i+j}$. Then roughly speaking, the result says that if $u \in \widehat{\Gamma}_1(v_0, w_0)$ and $J_1(u) < \infty$, u must get L^2 close to v_0 or w_0 at least for a sequence of sets X_i with $i \to \pm\infty$.

Proposition 6.27. *Suppose $(*)_0$ holds and $u \in \widehat{\Gamma}_1(v_0, w_0)$ with $J_1(u) \leq M < \infty$. Then for any $\sigma > 0$ and $t \in \mathbb{Z}$, there is an $\ell_0 = \ell_0(\sigma, M) \in \mathbb{N}$ independent of u and t such that whenever $\ell \in \mathbb{N}$ and $\ell \geq \ell_0$,*

$$\|u - \varphi\|_{L^2(X_i)} \leq \sigma \tag{6.28}$$

for some $i = i(\ell, t) \in (t - \ell + 2, t + \ell - 2)$ and $\varphi = \varphi_{\ell,t} \in \{v_0, w_0\}$.

Proof. If the proposition is false, there are a $\sigma > 0$, $t \in \mathbb{Z}$, and a sequence $(u_k) \subset \widehat{\Gamma}_1$ such that

$$J_1(u_k) \le M \tag{6.29}$$

and

$$\|u_k - \varphi\|_{L^2(X_i)} \ge \sigma \tag{6.30}$$

for $\varphi = v_0$ and w_0 and for all $i \in (t-k, t+k)$. By Lemma 2.22, (u_k) is bounded in $W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$. Hence there is a $U^* \in W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ such that along a subsequence, $u_k \to U^*$ weakly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$, strongly in $L^2_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$, and pointwise a.e. as $k \to \infty$. Therefore $U^* \in \widehat{\Gamma}_1$,

$$-K \le J_1(U^*) \le M + 2K \tag{6.31}$$

as in (3.6)–(3.7), and

$$\|U^* - \varphi\|_{L^2(X_i)} \ge \sigma \tag{6.32}$$

for all $i \in \mathbb{Z}$ and $\varphi \in \{v_0, w_0\}$.

To complete the proof, it suffices to show that such a U^* cannot exist.

Choose $U \in \mathcal{M}_1(v_0, w_0)$ as given by Theorem 3.2 and further require that

$$\|U - w_0\|_{L^2(X_0)} \le \frac{\sigma}{3}. \tag{6.33}$$

Set

$$\mathcal{B} = \{\tau^1_{-j} U^* \mid j \in \mathbb{Z}\}$$

and define

$$\begin{aligned} \mathcal{Y} = \big\{ & u \in \widehat{\Gamma}_1(v_0, w_0) \mid u \le U \quad \text{and} \\ & \|u - g\|_{L^2(T_i)} \to 0, \quad \text{as} \quad i \to \infty \quad \text{for some} \quad g = g(u) \in \mathcal{B}\big\}. \end{aligned}$$

Note that $\mathcal{Y}$ satisfies (Y^1_1) of Proposition 2.50. Setting

$$f = \begin{cases} v_0, & x_1 \le 0, \\ x_1 \min(U, U^*) + (1 - x_1) v_0, & 0 \le x_1 \le 1, \\ \min(U, U^*), & x_1 \ge 1, \end{cases}$$

shows $f \in \mathcal{Y} \ne \emptyset$. Thus if

$$c_1(\mathcal{Y}) = \inf_{u \in \mathcal{Y}} J_1(u), \tag{6.34}$$

by Proposition 2.8 and (6.34),

$$-K_1 \le c_1(\mathcal{Y}) \le J_1(f) < \infty. \tag{6.35}$$

Let (φ_k) be a minimizing sequence for (6.34). Then for each $k \in \mathbb{N}$, there are an $s_k \in \mathbb{N}$ and $g_k \in \mathcal{B}$ such that if $s \geq s_k$,

$$\|\varphi_k - g_k\|_{L^2(X_s)} \leq \frac{\sigma}{3}. \tag{6.36}$$

Note that $J_1(\varphi_k) = J_1(\tau^1_{-s_k}\varphi_k)$. Since $\tau^1_{-s_k}\varphi_k$ need not belong to $\mathcal{Y}$, $(\tau^1_{-s_k}\varphi_k)$ may not be a minimizing sequence in $\mathcal{Y}$ for J_1. However, $(\tau^1_{-s_k}\varphi_k)$ can be modified to produce such a minimizing sequence. This will be shown next.

Let $\psi_k = \max(\tau^1_{-s_k}\varphi_k, U)$ and $\chi_k = \min(\tau^1_{-s_k}\varphi_k, U)$. We claim that $\psi_k \in \Gamma_1(v_0, w_0)$ and $\chi_k \in \mathcal{Y}$. The only point that need be checked is the asymptotic behavior of the functions as $x_1 \to \infty$. We will show that

$$\|\chi_k - \tau^1_{-s_k} g_k\|_{L^2(T_i)} \to 0 \tag{6.37}$$

as $i \to \infty$. Indeed, observe that

$$\begin{aligned}\int_{T_i} |\chi_k - \tau^1_{-s_k} g_k|^2 dx \leq & \int_{T_i \cap \{U \geq \tau^1_{-s_k}\varphi_k\}} |\tau^1_{-s_k}\varphi_k - \tau^1_{-s_k} g_k|^2 dx \\ & + \int_{T_i \cap \{\tau^1_{-s_k} g_k \leq U < \tau^1_{-s_k}\varphi_k\}} |\tau^1_{-s_k}\varphi_k - \tau^1_{-s_k} g_k|^2 dx \\ & + \int_{T_i \cap \{U < \min(g_k, \tau^1_{-s_k}\varphi_k)\}} |U - w_0|^2 dx \\ \leq & \int_{T_{i+s_k}} |\varphi_k - g_k|^2 dx + \int_{T_i} |U - w_0|^2 dx \to 0 \end{aligned} \tag{6.38}$$

as $i \to \infty$. The asymptotics for ψ_k follow in a similar but simpler fashion.

Next we show that

$$J_1(\psi_k) + J_1(\chi_k) = J_1(\varphi_k) + J_1(U). \tag{6.39}$$

Expressions like (6.39) have been used several times earlier. They have always involved functions u for which $J_1(u) < \infty$. In general, as defined $J_1(u)$ is a lim inf but when $J_1(u) < \infty$, it has a simpler form as a limit. Equation (6.39) represents the first time we may actually encounter a lim inf. Thus more care is needed to verify (6.39) for this case. As earlier, for any $p < q \in \mathbb{Z}$,

$$J_{1;p,q}(\psi_k) + J_{1;p,q}(\chi_k) = J_{1;p,q}(\tau^1_{-s_k}\varphi_k) + J_{1;p,q}(U). \tag{6.40}$$

We can assume $J_1(\varphi_k) < \infty$. Therefore (6.40) and (2.23) imply $J_1(\psi_k), J_1(\chi_k) < \infty$. Choose $p_i \to -\infty$, $q_i \to \infty$ as $i \to \infty$ such that $J_{1;p_i,q_i}(\tau^1_{-s_k}\varphi_k) \to J_1(\varphi_k)$. This plus Proposition 2.24 for ψ_k and U gives us control of the limits of three of the terms in (6.40). Hence by (6.40) as $i \to \infty$, $J_{1;p_i,q_i}(\chi_k)$ converges to $\alpha \geq J_1(\chi_k)$. In fact, $\alpha = J_1(\chi_k)$, for otherwise $\alpha > J_1(\chi_k)$ and

$$J_1(\psi_k) + J_1(\chi_k) < J_1(\varphi_k) + J_1(U). \tag{6.41}$$

On the other hand, if $s_m^* \to -\infty$ and $t_m^* \to \infty$ as $m \to \infty$ and $J_{1;s_m^*,t*_m}(\chi_k) \to J_1(\chi_k)$, by (6.40),

$$J_1(\psi_k) + J_1(\chi_k) = \lim_{m\to\infty} J_{1;s_m,t_m}(\varphi_k) + J_1(U), \tag{6.42}$$

so by (6.41)–(6.42),

$$\lim_{m\to\infty} J_{1;s_m,t_m}(\varphi_k) < J_1(\varphi_k) = \varliminf_{\substack{p\to-\infty,\\ q\to\infty}} J_{1;p,q}(\varphi_k), \tag{6.43}$$

a contradiction. Therefore (6.39) is valid for the current setting.

Since $\psi_k \in \Gamma_1(v_0, w_0)$, (6.39) shows that

$$J_1(\chi_k) \leq J_1(\varphi_k). \tag{6.44}$$

Consequently, (χ_k) is a minimizing sequence for (6.34). Thus by Proposition 2.50, it can be assumed that there is a $\Phi \in \widehat{\Gamma}_1(v_0, w_0)$ such that $\chi_k \to \Phi$ in $W^{1,2}_{\text{loc}}(\mathbb{R}\times\mathbb{T}^{n-1})$ as $k \to \infty$. For any t, φ as in (Y_2^1) of Proposition 2.64

$$c_1(\mathcal{Y}) \leq J_1(\chi_k) \equiv c_1(\mathcal{Y}) + \delta_k \leq J_1(\chi_k + t\varphi) + \delta_k$$

with $\delta_k \to 0$ as $k \to \infty$. Therefore (Y_2^1) is satisfied, and by Proposition 2.64, Φ is a solution of (PDE) in $\mathbb{R}\times\mathbb{T}^{n-1}$.

We claim that for $i \geq 0$,

$$\|\Phi - w_0\|_{L^2(X_i)} \geq \frac{\sigma}{2}. \tag{6.45}$$

It suffices to show that

$$\|\chi_k - w_0\|_{L^2(X_i)} \geq \frac{\sigma}{2}. \tag{6.46}$$

By (6.32),

$$\begin{aligned}\|\chi_k - w_0\|_{L^2(X_i)} &\geq \|w_0 - \tau^1_{-s_k} g_k\|_{L^2(X_i)} - \|\chi_k - \tau^1_{-s_k} g_k\|_{L^2(X_i)} \\ &\geq \sigma - \|\chi_k - \tau^1_{-s_k} g_k\|_{L^2(X_i)}.\end{aligned} \tag{6.47}$$

Now as in (6.38),

$$\int_{X_i} |\chi_k - \tau^1_{-s_k} g_k|^2 dx \leq \int_{X_i \cap \{U \geq \tau^1_{-s_k} \varphi_k\}} |\tau^1_{-s_k} \varphi_k - \tau^1_{-s_k} g_k|^2 dx + \int_{X_i \cap \{g_k \leq U \leq \tau^1_{-s_k} \varphi_k\}} |\tau^1_{-s_k} \varphi_k - \tau^1_{-s_k} g_k|^2 dx + \int_{X_i \cap \{U < \min(g_k, \tau^1_{-s_k} \varphi_k)\}} |U - w_0|^2 dx. \quad (6.48)$$

Therefore by (6.36) and (6.33),

$$\|\chi_k - \tau^1_{-s_k} g_k\|^2_{L^2(X_i)} \leq \left(\frac{\sigma}{3}\right)^2 + \left(\frac{\sigma}{3}\right)^2 = \frac{2}{9}\sigma^2, \quad (6.49)$$

so (6.46) follows from (6.47) and (6.49).

Next choose $W \in \mathcal{M}_1(v_0, w_0)$ such that $W < \Phi$ in X_0. This is possible, provided that $\Phi > v_0$ in X_0. Assume for the moment that this is the case. Let

$$P_k = \min(W, \chi_k) \text{ and } Q_k = \max(W, \chi_k).$$

As above, $P_k \in \mathcal{Y}$ and $Q_k \in \Gamma_1(v_0, w_0)$. Therefore as in (6.44) and the lines that follow it,

$$J_1(P_k) \leq J_1(\chi_k), \quad (6.50)$$

and as $k \to \infty$, P_k converges in $W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ to a solution P of (PDE) with $P = \min(W, \Phi)$. Hence $P \leq W$, and by construction, $P = W < \Phi$ on X_0. Therefore the maximum principle argument of Theorem 3.2 yields $P \equiv W$. Consequently, $\|P - w_0\|_{L^2(X_i)} \to 0$ as $i \to \infty$. But $P \leq \Phi$, so by (6.45), $\|P - w_0\|_{L^2(X_i)} \geq \sigma/2$ for all $i \geq 0$. This contradiction shows that U^* cannot exist.

Now to complete the proof of Proposition 6.27, we must show that $\Phi > v_0$ in X_0. If not, $\Phi(z) = v_0(z)$ for some $z \in X_0$. Since $\Phi \in \hat{\Gamma}_1(v_0, w_0)$, $\Phi \geq v_0$. Thus Φ is a solution of (PDE) with a global minimum at z. Hence the maximum principle argument of Theorem 3.2 implies $\Phi \equiv v_0$. Therefore $\chi_k \to v_0$ in $L^2(X_0)$. On the other hand, $U > v_0$, so $\chi_k = \min(\tau^1_{-s_k} \varphi_k, U)$ implies

$$\|\tau^1_{-s_k} \varphi_k - v_0\|_{L^2(X_0)} = \|\varphi_k - v_0\|_{L^2(X_{s_k})} \to 0 \quad (6.51)$$

as $k \to \infty$. But by (6.32), (6.36), and (6.51),

$$\sigma \leq \|g_k - v_0\|_{L^2(X_{s_k})} \leq \|g_k - \varphi_k\|_{L^2(X_{s_k})} + \|\varphi_k - v_0\|_{L^2(X_{s_k})} \leq \frac{\sigma}{3} + \frac{\sigma}{3} \quad (6.52)$$

for large k, a contradiction. Thus the proof of Proposition 6.27 is complete.

The next result shows how Proposition 6.27 can provide asymptotic information about solutions of (PDE).

Proposition 6.53. *Under the hypotheses of Proposition 6.27, suppose there is an $R > 0$ such that u is a solution of (PDE) for $x_1 \geq R$ (resp. $x_1 \leq -R$). Then for some $\varphi \in \{v_0, w_0\}$, $\|u - \varphi\|_{W^{1,2}(X_i)} \to 0$ as $i \to \infty$ (resp. $\|u - \varphi\|_{W^{1,2}(X_i)} \to 0$ as $i \to -\infty$).*

A more refined conclusion is:

Corollary 6.54. *Under the hypotheses of Proposition 6.53, $\|u - \varphi\|_{C^2(T_i)} \to 0$ as $i \to -\infty$ (resp. $\|u - \varphi\|_{C^2(T_i)} \to 0$ as $i \to -\infty$).*

Proof of Proposition 6.53. Choose $\sigma > 0$ and free for the moment. Apply Proposition 6.27 to a sequence $t_k \to \infty$ to obtain a $\varphi \in \{v_0, w_0\}$ and a corresponding sequence $(s_k(\sigma)) \subset \mathbb{N}$ with $s_k(\sigma) \to \infty$ as $k \to \infty$ and such that

$$\|u - \varphi\|_{L^2(X_{s_k(\sigma)})} \leq \sigma. \tag{6.55}$$

With φ so determined, it suffices to show that

$$\|u - \varphi\|_{L^2(X_i)} \to 0, \quad i \to \infty. \tag{6.56}$$

Indeed, assuming (6.56) for now, we claim that there is a constant M_3 independent of u and i such that whenever $s_i > R + 2$,

$$\|u - \varphi\|_{W^{1,2}(Z_{s_i(\sigma)})} \leq M_3 \|u - \varphi\|_{L^2(X_{s_i(\sigma)})}, \tag{6.57}$$

where $Z_p = \bigcup_{j=-1}^{1} T_{p+j}$. To verify (6.57), set $\Phi = u - \varphi$. Then as in (2.5), Φ satisfies

$$-\Delta\Phi + A\Phi = 0. \tag{6.58}$$

Now following the argument from (4.69)–(4.71) shows that (6.57) follows from (4.71). Moreover, (6.56) and (6.57) imply the proposition.

It remains only to verify (6.56). If it is false, there is are $\gamma > 0$ and a sequence $p_i \to \infty$ as $i \to \infty$ such that

$$\|u - \varphi\|_{L^2(X_{p_i})} \geq \gamma. \tag{6.59}$$

Relabeling $s_k(\sigma)$, it can be assumed that $p_i \in (s_i(\sigma), s_{i+1}(\sigma))$. Define

$$f_i = \begin{cases} u, & x_1 \leq s_i - 1, \\ \varphi, & s_i \leq x_1 \leq s_i + 1, \\ u, & s_i + 2 \leq x_1 \leq s_{i+1} - 1, \\ \varphi, & s_{i+1} \leq x_1 \leq s_{i+1} + 1, \\ u, & s_{i+1} + 2 \leq x_1, \end{cases} \tag{6.60}$$

with the usual interpolation in between. Then as in (3.23), there is a $\kappa(\sigma)$ with $\kappa(\sigma) \to 0$ as $\sigma \to 0$ such that

$$|J_{1;s_i,s_{i+1}-1}(u) - J_{1;s_i,s_{i+1}-1}(f_i)| \le \kappa(\sigma). \tag{6.61}$$

Set

$$h_i = \begin{cases} \varphi, & x_1 \le s_i, \\ f_i, & s_i \le x_1 \le s_{i+1}, \\ \varphi, & s_{i+1} \le x_1. \end{cases}$$

Then $h_i \in \Gamma_1(\varphi)$, so by (6.59) and Proposition 6.13,

$$J_1(h_i) \ge \beta(\gamma). \tag{6.62}$$

Since

$$J_1(h_i) = J_{1;s_i,s_{i+1}-1}(f_i), \tag{6.63}$$

by (6.61)–(6.63),

$$J_{1;s_i,s_{i+1}-1}(u) \ge \beta(\gamma) - \kappa(\sigma). \tag{6.64}$$

Choose σ so small that

$$2\kappa(\sigma) \le \beta(\gamma). \tag{6.65}$$

Therefore (6.64) becomes

$$J_{1;s_i,s_{i+1}-1}(u) \ge \frac{1}{2}\beta(\gamma). \tag{6.66}$$

Now suppose that $s_i > R + 2$ for $i \ge i_0$. With $q \in \mathbb{N}$ free for the moment, write

$$J_1(u) = J_{1;-\infty,s_{i_0}-1}(u) + \sum_{j=0}^{q-1} J_{1;s_{i_0+j},s_{i_0+j+1}-1}(u) + J_{1;s_{i_0+q},\infty}(u). \tag{6.67}$$

Since by hypothesis, $J_1(u) \le M$, by (6.66)–(6.67) and Lemma 2.22,

$$M + 2K_1 \ge \frac{q}{2}\beta(\gamma). \tag{6.68}$$

But (6.68) is not possible for large q. Thus (6.56) holds, and Proposition 6.53 is proved.

Proof of Corollary 6.54. Observe that with $\Phi = u - \varphi$ as in (6.58), $|\Phi| \le 1$. Hence by the L^p_{loc} elliptic estimates for (6.58) with $p > 2$, for any $z \in T_i$ and $i > R + 2$,

$$\|\Phi\|_{W^{2,p}(B_1(z))} \le M_4 \|\Phi\|_{L^p(B_2(z))} \le M_4 \|\Phi\|^{2/p}_{L^2(B_2(z))} \le M_4 \|\Phi\|^{2/p}_{L^2(X_i)}, \tag{6.69}$$

with M_4 a constant independent of u, i, and $z \in T_i$. Thus for $p > \frac{n}{2}$, (6.69), the Sobolev embedding theorem, and Proposition 6.53 imply

$$\|\Phi\|_{C^1(T_i)} \to 0, \quad i \to \infty. \tag{6.70}$$

By the interior Schauder estimates, for any $\alpha \in (0, 1)$, there is a constant M_5 such that

$$\|\Phi\|_{C^{2,\alpha}(B_1(z))} \le M_5 \tag{6.71}$$

for all $z \in [R + 2, \infty) \times \mathbb{T}^{n-1}$. Now (6.70)–(6.71) and standard interpolation inequalities yield

$$\|\Phi\|_{C^2(T_i)} \to 0, \quad i \to \infty. \tag{6.72}$$

One final comparison result is needed to prove Theorem 6.8. With ρ_i as in (6.2), define

$$\Lambda_1(v_0, w_0) = \left\{u \in \Gamma_1(v_0, w_0) \mid \|u - v_0\|_{L^2(T_0)} = \rho_1 \text{ or } \|u - w_0\|_{L^2(T_0)} = \rho_2\right\}$$

and

$$d_1(v_0, w_0) = \inf_{u \in \Lambda_1(v_0, w_0)} J_1(u). \tag{6.73}$$

Replacing ρ_1 by ρ_4 and ρ_2 by ρ_3, $\Lambda_1(w_0, v_0)$ and $d_1(w_0, v_0)$ are defined similarly.

Proposition 6.74. *$d_1(v_0, w_0) > c_1(v_0, w_0)$ and $d_1(w_0, v_0) > c_1(w_0, v_0)$.*

Proof. Their proofs being the same, only the first inequality will be proved. Since $\Lambda_1(v_0, w_0) \subset \Gamma_1(v_0, w_0)$,

$$d_1(w_0, v_0) \ge c_1(w_0, v_0). \tag{6.75}$$

To exclude equality in (6.75), let (u_k) be a minimizing sequence for (6.73). By Propositions 2.50 and 2.64, it can be assumed that there is a $P \in \widehat{\Gamma}_1(v_0, w_0)$ with $J_1(P) < \infty$ such that $u_k \to P$ in $W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$,

$$\|P - v_0\|_{L^2(T_0)} = \rho_1 \quad \text{or} \quad \|P - w_0\|_{L^2(T_0)} = \rho_2 \tag{6.76}$$

and P is a solution of (PDE) whenever $x \notin [0, 1] \times \mathbb{T}^{n-1}$. Moreover, Proposition 6.53 applies to P, so

$$\|P - \varphi\|_{W^{1,2}(X_i)}, \quad \|P - \psi\|_{W^{1,2}(X_{-i})} \to 0 \tag{6.77}$$

as $i \to \infty$ for some $\varphi, \psi \in \{v_0, w_0\}$. Suppose, e.g., $\psi = w_0$. Choose $\varepsilon > 0$. Then there is an $s \in -\mathbb{N}$ such that for all $k \geq k_0(s)$,

$$\|u_k - w_0\|_{W^{1,2}(X_s)} \leq \varepsilon. \tag{6.78}$$

Since $u_k \in \Gamma_1(v_0, w_0)$, for any $q = q(k) \in \mathbb{N}$ and sufficiently large,

$$\|u_k - w_0\|_{W^{1,2}(X_q)} \leq \varepsilon. \tag{6.79}$$

Define

$$f_k = \begin{cases} u_k, & x \notin Z_s \cup Z_q, \\ w_0, & x \in T_s \cup T_q, \end{cases} \tag{6.80}$$

with the usual interpolation in the remaining four regions. Then as for (6.61), there is a function $\kappa(\theta)$ with $\kappa(\theta) \to 0$ as $\theta \to 0$ such that

$$|J_1(u_k) - J_1(f_k)| \leq \kappa(\varepsilon). \tag{6.81}$$

Set

$$g_k = \begin{cases} w_0, & x_1 \leq s, \\ f_k, & s \leq x_1 \leq q+1, \\ w_0, & q+1 \leq x_1, \end{cases} \tag{6.82}$$

and

$$h_k = \begin{cases} f_k, & x_1 \leq s, \\ w_0, & s \leq x_1 \leq q+1, \\ f_k, & q+1 \leq x_1, \end{cases} \tag{6.83}$$

so

$$J_1(f_k) = J_1(g_k) + J_1(h_k). \tag{6.84}$$

Moreover, $g_k \in \Gamma_1(w_0)$ and

$$\|g_k - w_0\|_{L^2(T_0)} = \|u_k - w_0\|_{L^2(T_0)}. \tag{6.85}$$

Thus either

$$\|P - w_0\|_{L^2(T_0)} = \rho_2,$$

in which case it can be assumed that

$$\|u_k - w_0\|_{L^2(T_0)} = \rho_2, \tag{6.86}$$

or

$$\|P - v_0\|_{L^2(T_0)} = \rho_1,$$

in which case

$$\|u_k - w_0\|_{L^2(T_0)} \geq \overline{\rho} - \rho_1. \tag{6.87}$$

Recall $\overline{\rho} = \|w_0 - v_0\|_{L^2(T_0)}$. Thus by (6.85)–(6.87),

$$\|w_0 - g_k\|_{L^2(T_0)} \geq \min(\rho_2, \overline{\rho} - \rho_1) \equiv \gamma, \tag{6.88}$$

so by Proposition 6.13,

$$J_1(g_k) \geq \beta(\gamma). \tag{6.89}$$

Since $h_k \in \Gamma_1(v_0, w_0)$, by (6.81), (6.84), and (6.89),

$$J_1(u_k) \geq -\kappa(\varepsilon) + \beta(\gamma) + c_1(v_0, w_0). \tag{6.90}$$

Choose ε so small that

$$2\kappa(\varepsilon) \leq \beta(\gamma) \tag{6.91}$$

and let $k \to \infty$ in (6.90), yielding

$$d_1(v_0, w_0) \geq c_1(v_0, w_0) + \frac{1}{2}\beta(\gamma). \tag{6.92}$$

If $\varphi = v_0$, a similar argument gives (6.92) with γ replaced by $\min(\rho_1, \overline{\rho} - \rho_2)$. One case remains: $\psi = v_0$ and $\varphi = w_0$. Then $P \in \Lambda_1(v_0, w_0)$ and therefore $J_1(P) \geq d_1(w_0, v_0)$. An argument essentially as in the proof of Theorem 3.2, in particular the proof of (C) beginning with (3.15), shows that $J_1(P) = d_1(v_0, w_0)$. If $d_1(v_0, w_0) = c_1(v_0, w_0)$, the fact that $P \in \Gamma_1(v_0, w_0)$ and 2^o of Theorem 3.2 show that P is a solution of (PDE). But P satisfies (6.76), which is incompatible with (6.2). Thus $d_1(v_0, w_0) > c_1(v_0, w_0)$ for all three cases, and Proposition 6.74 is proved.

For the final result in this section we give a partial answer to a question posed by Moser [1] and by Bangert [2]. They noted that for $n = 1$, if u is minimal, then u is WSI. They asked what further conditions one needs for u to be WSI when $n > 1$. Some sufficient conditions were given by Bangert in [2]. The next result provides another partial answer of a different spirit from those of [2]. We thank Sergey Bolotin for a helpful suggestion.

Proposition 6.93. *Suppose $u \in \widehat{\Gamma}_1(v_0, w_0)$, and u is minimal. Then u is WSI.*

Proof. Since u is minimal, it is a solution of (PDE). Suppose for the moment that $J_1(u) < \infty$. Then by Corollary 6.54, $\|u - \varphi\|_{C^2(T_i)} \to 0$ as $i \to -\infty$ and $\|u - \psi\|_{C^2(T_i)} \to 0$ as $i \to \infty$, where $\varphi, \psi \in \{v_0, w_0\}$. We consider two cases: (a) $\varphi = \psi$ and (b) $\varphi \neq \psi$. For (a), $u \in \Gamma_1(v_0) \cup \Gamma_1(w_0)$. Let $p \in \mathbb{N}$ and set $u_p = \varphi$, $|x_1| \leq p$; $u_p = u, |x_1| \geq p + 1$; and interpolate as usual for $p < |x_1| < p + 1$. Then by the minimality of u,

$$J_1(u) \leq J_1(u_p) \to 0, \quad p \to \infty. \tag{6.94}$$

Thus by Theorem 2.72, $J_1(u) = 0$ and $u = \varphi$. Consequently u is WSI. Similarly for (b), $u \in \Gamma_1(v_0, w_0) \cup \Gamma_1(w_0, v_0)$. We claim that $u \in \mathcal{M}_1(v_0, w_0) \cup \mathcal{M}_1(w_0, v_0)$ and therefore u is WSI via 2^o(c) of Theorem 3.2. If the claim is false, say $u \in \Gamma_1(v_0, w_0)$, then

$$J_1(u) > c_1(v_0, w_0). \tag{6.95}$$

Let $U \in \mathcal{M}_1(v_0, w_0)$. As for case (a), set $U_p = U$, $|x_1| \leq p$; $U_p = u$, $|x_1| \geq p + 1$; and interpolate for $p < |x_1| < p + 1$. Then

$$J_1(u) \leq J_1(U_p) \to c_1(v_0, w_0), \quad p \to \infty, \tag{6.96}$$

so $J_1(u) \leq c_1(v_0, w_0)$, contrary to (6.95). Thus $u \in \mathcal{M}_1(v_0, w_0)$.

It remains to prove that $J_1(u) < \infty$. Let u_p be defined as in case (a) with $\phi = w_0$. Then

$$0 \leq \sum_{-p-1}^{p} \left\{J_{1,i}(u_p) - J_{1,i}(u)\right\} = J_1(u_p) - J_1(u). \tag{6.97}$$

Since u is a solution of (PDE), $|J_{1,-p-1}(u_p)|$ and $|J_{1,p}(u_p)|$ are bounded by a constant K depending only on u. Therefore by (6.97),

$$\sum_{-p-1}^{p} J_{1,i}(u) \leq 2K$$

and letting $p \to \infty$ shows $J_1(u) < \infty$.

Chapter 7
The Proof of Theorem 6.8

The proof consists of several steps. Let (u_k) be a minimizing sequence for (6.7). Thus there is an $M > 0$ such that

$$J_1(u_k) \le M \tag{7.1}$$

for all $k \in \mathbb{N}$. In fact, if $V_1 \in \mathcal{M}_1(v_0, w_0)$ and $W_1 \in \mathcal{M}_1(w_0, v_0)$ such that V_1 satisfies (6.5) (i) and W_1 satisfies (6.5) (iv), setting

$$\widehat{U} = \begin{cases} V_1, & x_1 \le m_1, \\ w_0, & m_1 + 1 \le x_1 \le m_4 - 1, \\ W_1, & m_4 \le x_1, \end{cases}$$

with the usual interpolation in between, $J_1(\widehat{U})$ furnishes an upper bound for $J_1(u_k)$ independently of m and ℓ. The set $Y_{m,\ell}$ satisfies (Y_1^1), so by Proposition 2.50, it can be assumed that there is a $U \in \widehat{\Gamma}_1(v_0, w_0)$ such that (u_k) converges to U in $W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$. Therefore U satisfies (6.5). As in (3.6)–(3.7),

$$J_1(U) \le M + 2K_1. \tag{7.2}$$

Moreover, as in the proof of Theorem 3.2, U is a solution of (PDE) except possibly for the four integral constraint regions.

The remainder of the proof is divided as follows: We show (A) for ℓ sufficiently large, there is an X_i in each integral constraint region such that U satisfies (PDE) in the interior of X_i; (B) U satisfies (6.6) and therefore $U \in Y_{m,\ell}$; (C) $J_1(U) = b_{m,\ell}$; (D) for $m_2 - m_1$ and $m_4 - m_3$ sufficiently large, U satisfies (PDE) in the integral constraint regions; (E) U satisfies (6.9).

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Proof of (A). Choose σ so that

$$0 < \sigma < \min_{1 \le j \le 4} (\rho_j, \bar{\rho} - \rho_j). \tag{7.3}$$

It can be assumed that $\ell \ge \ell_0(\sigma, M)$ with $M = J_1(\widehat{U})$ and ℓ_0 given by Proposition 6.27. Let $\mathcal{R}$ be any of the integral constraint regions. Then by Proposition 6.27, there are an $X_i \subset \mathcal{R}$ and $\varphi_i \in \{v_0, w_0\}$ such that

$$\|U - \varphi_i\|_{L^2(X_i)} \le \sigma. \tag{7.4}$$

The choice of σ in (7.3) implies $\varphi_i = v_0$ if $\mathcal{R} = \mathcal{R}_1 \equiv [m_1 - \ell, m_1] \times \mathbb{T}^{n-1}$ or $\mathcal{R}_4 \equiv [m_4, m_4 + \ell] \times \mathbb{T}^{n-1}$ and $\varphi_i = w_0$ if $\mathcal{R} = \mathcal{R}_2 \equiv [m_i, m_i + \ell] \times \mathbb{T}^{n-1}$ or $\mathcal{R}_3 \equiv [m_3 - \ell, m_3] \times \mathbb{T}^{n-1}$. For example, if $X_i \subset \mathcal{R}_2$ and $\varphi_i = v_0$, by (7.4) and (6.5) (ii),

$$\sigma \ge \|U - v_0\|_{L^2(X_i)} \ge \|U - v_0\|_{L^2(T_i)} \ge \bar{\rho} - \|U - w_0\|_{L^2(T_i)} \ge \bar{\rho} - \rho_2, \tag{7.5}$$

contrary to (7.3). The remaining cases are proved similarly. Thus (7.3) shows that U satisfies the integral constraint for these special X_i's with strict inequality. Thus so does u_k for large k. Hence for $z \in X_i$ and r sufficiently small, the proof of (A) of Theorem 3.2 shows that U is a solution of (PDE) in the interior of X_i.

Proof of (B). To obtain (6.6), note first that by Proposition 6.53 with $R = m_4 + \ell$,

$$\|U - \varphi\|_{L^2(X_j)} \to 0, \quad j \to \infty, \tag{7.6}$$

for some $\varphi \in \{v_0, w_0\}$. If $\varphi = v_0$, then (B) is proved. Otherwise, $\varphi = w_0$. Then by (7.6), for some $p > m_4 + \ell$,

$$\|U - v_0\|_{L^2(T_p)} \ge \frac{3}{4}\bar{\rho}, \tag{7.7}$$

and by the convergence of u_k to U in $W^{1,2}(T_p)$,

$$\|u_k - v_0\|_{L^2(T_p)} \ge \frac{1}{2}\bar{\rho} \tag{7.8}$$

for all large k. By Proposition 6.27 and the argument of (A), there is an $i \in (m_4 + 2, m_4 + \ell - 2)$ such that

$$\|U - v_0\|_{L^2(X_i)} \le \sigma. \tag{7.9}$$

Since U and v_0 are solutions of (PDE) in X_i, as in (4.68)–(4.71),

$$\|U - v_0\|_{W^{1,2}(Z_i)} \le M_3\sigma. \tag{7.10}$$

Hence for large k, the $W^{1,2}_{\text{loc}}$ convergence of u_k to U yields

$$\|u_k - v_0\|_{W^{1,2}(T_i)} \le 2M_3\sigma. \tag{7.11}$$

Now a cutting and pasting argument as in the proof of Proposition 6.53 will establish (6.6). Choose $q_k > p$ so that

$$\|u_k - v_0\|_{W^{1,2}(X_{q_k})} \le M_3\sigma. \tag{7.12}$$

As in (6.60), define

$$f_k = \begin{cases} u_k, & x_1 \le i-1, \\ v_0, & i \le x_1 \le i+1, \\ u_k, & i+2 \le x_1 \le q_k - 1, \\ v_0, & q_k \le x_1 \le q_k + 1, \\ u_k, & q_k + 2 \le x_1, \end{cases} \tag{7.13}$$

with the usual interpolation otherwise. Then as in (6.61),

$$|J_{1;i,q_k}(u_k) - J_{1;i,q_k}(f_k)| \le \kappa(\sigma) \tag{7.14}$$

with $\kappa(\theta) \to 0$ as $\theta \to 0$.

Define

$$h_k = \begin{cases} v_0, & x_1 \le i, \\ f_k, & i \le x_1 \le q_k, \\ v_0, & q_k \le x_1. \end{cases} \tag{7.15}$$

Thus $h_k \in \Gamma_1(v_0)$, and by (7.15) and (7.8),

$$J_1(h_k) = J_{1;i,q_k}(f_k) \ge \beta(\bar{\rho}/2), \tag{7.16}$$

β being given by Proposition 6.13. Now by (7.14) and (7.16),

$$J_1(u_k) \ge J_{1;-\infty,i-1}(u_k) + \beta(\bar{\rho}/2) - \kappa(\sigma) + J_{1;q_k+1,\infty}(u_k). \tag{7.17}$$

But setting

$$g_k = \begin{cases} u_k, & x_1 \le i-1, \\ v_0, & i \le x_1 \le q_k + 1, \\ u_k, & q_k + 2 \le x_1, \end{cases} \tag{7.18}$$

and interpolating in between as usual, it can be assumed that

$$|J_{1;i-1}(u_k) - J_{1;i-1}(g_k)| + |J_{1,q_k+1}(u_k) - J_{1;q_k+1}(g_k)| \le \kappa(\sigma). \tag{7.19}$$

Therefore (7.18)–(7.19) show that

$$J_{1;-\infty,i-1}(u_k) + J_{1;q_{k+1},\infty}(u_k) \geq J_1(g_k) - \kappa(\sigma). \tag{7.20}$$

Combining (7.17) and (7.20) gives

$$J_1(u_k) \geq J_1(g_k) + \beta(\bar{\rho}/2) - 2\kappa(\sigma). \tag{7.21}$$

Choose σ so small that

$$4\kappa(\sigma) < \beta(\bar{\rho}/2). \tag{7.22}$$

But then, since $(g_k) \subset Y_{m,\ell}$, (7.21)–(7.22) show that (u_k) is not a minimizing sequence for (6.7). Thus (6.6) holds as $i \to \infty$, and a similar argument establishes (6.6) as $i \to -\infty$.

Proof of (C). By (B), $U \in Y_{m,\ell}$, so

$$J_1(U) \geq b_{m,\ell}. \tag{7.23}$$

The reverse inequality now follows exactly as in the proof of (C) of Theorem 3.2.

Proof of (D). As was shown in (A), whenever U satisfies one of the integral constraints with strict inequality, it is a solution of (PDE) in the interior of the corresponding T_i. Moreover, once it is known that there is strict inequality for all of the constraint regions (or even a pair of adjacent ones), the argument of (A) also shows that U is a solution of (PDE) at the associated boundary points. Thus to prove (D), it suffices to verify that there is strict inequality in (6.5) with $u = U$ for each region. This will be shown for (6.5) (i)–(ii), the remaining cases being treated similarly.

Suppose for some i in (6.5) (i)–(ii) there is equality. Then

$$\|U - \varphi\|_{L^2(T_i)} = \rho, \tag{7.24}$$

where $(\varphi, \rho) = (v_0, \rho_1)$ or (w_0, ρ_2). Using Proposition 6.27 and (6.57) again, there is a $q \in [m_3 - \ell + 2, m_3 - 3] \cap \mathbb{Z}$ such that

$$\|U - w_0\|_{W^{1,2}(X_q)} \leq M_3\sigma. \tag{7.25}$$

Define U^* via

$$U^* = \begin{cases} U, & x_1 \leq q - 1, \\ w_0, & q \leq x_1 \leq q + 1, \\ U, & q + 2 \leq x_1, \end{cases} \tag{7.26}$$

and interpolate as usual elsewhere. Then as in (4.71), by (7.25) and (7.26), there is a function $\kappa(\theta)$ with $\kappa(\theta) \to 0$ as $\theta \to 0$ such that

$$|J_1(U) - J_1(U^*)| \le \kappa(\sigma). \tag{7.27}$$

Define

$$\Phi = \begin{cases} U^*, & x_1 \le q+1, \\ w_0, & q+1 \le x_1, \end{cases} \tag{7.28}$$

and

$$\Psi = \begin{cases} w_0 & x_1 \le q, \\ U^*, & q \le x_1, \end{cases} \tag{7.29}$$

Note that $\tau_q^1 \Phi \in \Lambda_1(v_0, w_0)$. Therefore by Proposition 6.74,

$$J_1(\Phi) = J_1(\tau_q^1 \Phi) \ge d_1(v_0, w_0). \tag{7.30}$$

Since $\Psi \in \Gamma_1(v_0, w_0)$,

$$J_1(\Psi) \ge c_1(w_0, v_0). \tag{7.31}$$

Observing that

$$J_1(U^*) = J_1(\Phi) + J_1(\Psi), \tag{7.32}$$

by (7.30)–(7.32) and (7.27) we have:

$$J_1(U) \ge d_1(v_0, w_0) + c_1(w_0, v_0) - \kappa(\sigma). \tag{7.33}$$

On the other hand, an upper bound can be obtained for $J_1(U)$ since it is a minimizer of J_1 in $Y_{m.l}$. For $m_2 - m_1$ and $m_4 - m_3$ sufficiently large and any $\varepsilon = \varepsilon(m_2 - m_1, m_4 - m_3) > 0$, we can find $V_1 \in \mathcal{M}(v_0, w_0)$ and $W_1 \in \mathcal{M}(w_0, v_0)$ such that if

$$\hat{U} = \begin{cases} V_1, & x_1 \le q-1, \\ w_0, & q \le x_1 \le q+1, \\ W_1, & q+2 \le x_1, \end{cases} \tag{7.34}$$

then

$$J_1(U) \le J_1(\hat{U}) \le c_1(v_0, w_0) + c_1(w_0, v_0) + \varepsilon. \tag{7.35}$$

Now combining (7.33)–(7.35) yields

$$d_1(v_0, w_0) - c_1(v_0, w_0) \le \varepsilon + \kappa(\sigma). \tag{7.36}$$

Finally choosing ε and σ so small that

$$2(\varepsilon + \kappa(\sigma)) < \min(d_1(v_0, w_0) - c_1(v_0, w_0)) \tag{7.37}$$

holds shows (7.36) and (7.37) are not compatible. Thus we have a contradiction and (D) is proved.

Proof of (E). Since $\|U - v_0\|_{W^{1,2}(X_p)} \to 0$ as $|p| \to \infty$, via Proposition 2.24, the C^2 convergence follows from Corollary 6.54.

The proof of Theorem 6.8 is complete.

Remark 7.38. For an instructive geometrical example that illustrates Theorem 6.8 as well as Theorem 3.2, set $n = 1$, so (PDE) describes the motion of a nonlinear pendulum with x_1 becoming a time variable, t. Suppose that $F(t,z) \geq 0$ and $F(t,z) = 0$ if and only if $z \in \mathbb{Z}$. Then $\mathcal{M}_0 = \mathbb{Z}$, and $v_0 = 0, w_0 = 1$ is a gap pair. Changing variables so that v_0, w_0 become $-\pi, \pi$, these solutions represent a pendulum in a vertically upright position. Any member of $\mathcal{M}_1(v_0, w_0)$ starts at v_0 at $t = -\infty$ and rotates counterclockwise in a 1-monotone fashion, ending at w_0 at $t = \infty$. Similarly, any solution U of (PDE) in $Y_{m,l}$ represents a pendulum motion starting at $-\pi$, approaching π, and remaining near and below it for a time interval depending on $m_3 - m_2$ until finally returning to π at $t = \infty$.

Remark 7.39. The solution U of (PDE) given by Theorem 6.8 depends on $\ell \in \mathbb{N}$, $m \in \mathbb{Z}^4$ as well as on ρ_i, $1 \leq i \leq 4$. Letting $\ell, m_2 - m_1, m_4 - m_3 \to \infty$ shows that there are infinitely many distinct two transition solutions for any fixed set of ρ_i's. What is a minimal set of parameters that determine such solutions and how to give a more precise count of the number of distinct solutions remain interesting open questions.

The sets $\mathcal{M}_1(v_0, w_0)$ and $\mathcal{M}_1(w_0, v_0)$ are ordered. Fixing the ρ_i, $1 \leq i \leq 4$, the set of two transition solutions in $\bigcup_{m,\ell} Y_{m,\ell}(v_0, w_0)$ is certainly not ordered. For example $u \in Y_{m,\ell}(v_0, w_0)$ implies $\tau_{-1}^1 u \in Y_{m^*,\ell}(v_0, w_0)$, where $m^* = m + (1,1,1,1)$ and u and $\tau_{-1}^1 u$ must intersect. However, the next result shows that there are ordered pairs (and similarly ordered sequences) of solutions of (PDE) in $\bigcup_{\ell,m} Y_{m,\ell}(v_0, w_0)$.

Corollary 7.40. *Suppose (ℓ, m) and $(\bar{\ell}, \bar{m}) \in \mathbb{N} \times \mathbb{Z}^4$ satisfy the hypotheses of Theorem 6.8 for the same set of ρ_i's, $1 \leq i \leq 4$. Let $U_{m,\ell}$ be a solution of (PDE) corresponding to (ℓ, m). If also*

$$\bar{m}_2 + \bar{\ell} \ll m_1 - \ell; m_4 + \ell \ll \bar{m}_3 - \bar{\ell}, \tag{7.41}$$

then there is a solution $U_{\bar{m},\bar{\ell}} \in Y_{\bar{m},\bar{\ell}}$ of (PDE) such that $U_{\bar{m},\bar{\ell}} > U_{m,\ell}$.

Proof. A construction following the same lines as the proof of Theorem 6.8 will be employed. Set

$$Y(U_{m,\ell}) = \{u \in Y_{\bar{m},\bar{\ell}} \mid U_{m,\ell} \leq u\}.$$

By (7.41), $Y(U_{m,\ell}) \neq \emptyset$. Define

$$c(Y(U_{m,\ell})) = \inf_{u \in Y(U_{m,\ell})} J_1(u). \tag{7.42}$$

Let (u_k) be a minimizing sequence for (7.42). Then as in the proof of Theorem 6.8, there is a $U \in W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ such that along a subsequence, $u_k \to U$ in $W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$. Therefore $U \geq U_{m,\ell}$, and it satisfies the requirements for membership in $Y(U_{m,\ell})$ aside possibly from the asymptotic conditions (6.6). Moreover, U is a solution of (PDE) in any set T_i if this set does not involve an integral constraint. To see this, arguing as in (3.8)–(3.12) of the proof of Theorem 3.2 with v_0 replaced by $U_{m,\ell}$ shows that (Y_2^1) holds for any such T_i. Therefore U is a solution of (PDE) in T_i. Next, following (A)–(D) of the proof of Theorem 6.8 and the argument of (3.8)–(3.12) shows successively that (a) U satisfies (PDE) in some X_i for each of the four integral constraint regions, (b) U satisfies the asymptotic conditions (6.6) and hence $U \in Y(U_{m,\ell})$, (c) $J_1(U) = c(Y(U_{m,\ell}))$, and (d) U is a solution of (PDE) in the remaining integral constraint regions. In particular, v_0 is replaced by $U_{m,\ell}$ in (7.13), (7.15), and (7.18). By construction, $U \geq U_{m,\ell}$ and a familiar maximum principle argument gives strict inequality.

Chapter 8
k-Transition Solutions for $k > 2$

The methods of Chapter 7 can be extended to construct multitransition solutions of (PDE) for $k > 2$. These solutions will be heteroclinic in x_1 from v_0 to w_0 (or from w_0 to v_0) and periodic in $x_2, \dots, x_n$ if k is odd while if k is even, they will be homoclinic to v_0 (or to w_0) in x_1 and periodic in $x_2, \dots, x_n$. For example, to get k-transition solutions, let $m \in \mathbb{Z}^{2k}$, $k > 2$, with $m_{i+1} > m_i$ and $m_i + 2\ell < m_{i+1}$ for even i. Choose numbers $\rho_i \in (0, \bar{\rho})$, $1 \le i \le 2k$, with ρ_i as in (6.2) for $1 \le i \le 4$ and $\rho_{i+4} = \rho_i$. If k is even, define $Y_{m,\ell}$ as in (6.4) with (6.5) replaced by the analogous $2k$ constraints. If k is odd, $Y_{m,\ell} = Y_{m,\ell}(v_0, w_0)$, and the asymptotic condition at $x_1 = \infty$ becomes $\|\tau_{-i}^1 u - w_0\|_{L^2(T_i)} \to 0$ as $i \to \infty$.

The theorem one obtains is

Theorem 8.1. *Suppose F satisfies (F_1)–(F_2), $k \ge 2$, and $(*)_0$ and suppose as well that $(*)_1$ for $\mathcal{M}_1(v_0, w_0)$ and $\mathcal{M}_1(w_0, v_0)$ are satisfied. If $\ell \gg 0$, there is a $U \in Y_{m,\ell}$ such that $J_1(U) = b_{m,\ell} \equiv \inf_{Y_{m,\ell}} J_1$. If also $m_2 - m_1, \dots, m_{2k} - m_{2k-1} \gg 0$, U is a solution of (PDE) and $\|U - v_0\|_{C^2(T_i)} \to 0$ as $i \to -\infty$, $\|U - \varphi\|_{C^2(T_i)} \to 0$ as $i \to \infty$ where $\varphi = v_0$ if k is even and $\varphi = w_0$ if k is odd.*

The proof of Theorem 8.1 is essentially the same as that of Theorem 6.8. Therefore the details will be omitted. We turn instead to the following question: are there are solutions of (PDE) with an infinite number of transitions between v_0 and w_0? There are three cases one can consider: (i) $m = (m_k)_{k \in \mathbb{N}}$ with $m_k \to \infty$ as $k \to \infty$; (ii) $m = (m_k)_{k \in -\mathbb{N}}$ with $m_k \to -\infty$ as $k \to -\infty$; (iii) $m = (m_k)_{k \in \mathbb{Z}}$, with $m_k \to -\infty$ as $k \to -\infty$ and $m_k \to \infty$ as $k \to \infty$. Case (i) corresponds to solutions asymptotic to v_0 (or w_0) as $x_1 \to -\infty$ and case (ii) to solutions asymptotic to v_0 (or w_0) as $x_1 \to \infty$. A natural approach to any of these cases is the following: truncate m, i.e., replace m by $m_j^* = \{m_s \mid |s| \le 2j\}$. Then invoke Theorem 8.1 to get a solution $u_j^* \in Y_{m_j^*,\ell}$ of (PDE) for each $j \in \mathbb{N}$. Since $v_0 \le u_j^* \le w_0$, one can use the L^p and Schauder elliptic theories to get $C^{2,\alpha}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ estimates for u_j^* independently of j. This permits us to pass to a limit along a subsequence as $j \to \infty$ to find a solution U^* of (PDE) satisfying the constraints (6.5) associated with m. There are two possible obstacles to carrying out this program. The lesser

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one is to show that for cases (i) and (ii), U^* has the appropriate asymptotic behavior. A more serious difficulty is in applying Theorem 8.1 to find u_j^*. That result requires $\ell, m_{2i} - m_{2i-1} \gg 0$ and a priori ℓ and the difference in the m_i's will depend on j and possibly go to ∞ as $j \to \infty$.

Rather than pursue this point, we will carry out another more geometrical approach in the spirit of [7] and [8] (see also [27]). This new approach employs Theorem 6.8 and enables us to find k- and ∞-transition solutions of (PDE) with equal facility. To begin, choose (ℓ, m), $(\bar{\ell}, \bar{m}) \in \mathbb{N} \times \mathbb{Z}^4$ so that there are solutions V in $Y_{m,\ell}(v_0, w_0)$ and $W \in Y_{\bar{m},\bar{\ell}}(w_0, v_0)$ given by Theorem 6.8. Since $V < w_0$, $W > v_0$, and V, W (with v_0 replaced by w_0) satisfy (6.9), there is a constant $\beta_0 > 0$ such that for all $x \in \mathbb{R} \times \mathbb{T}^{n-1}$,

$$w_0(x) - V(x), W(x) - v_0(x) \geq 2\beta_0. \tag{8.2}$$

Hence if V and W are sufficiently separated in the sense that

$$\begin{aligned} &\text{(i)}\ \ \bar{m}_1 - \bar{\ell} - v_0 > m_4 + \ell + v_0 \ \text{ or} \\ &\text{(ii)}\ \ m_1 - \ell - v_0 > \bar{m}_4 + \bar{\ell} + v_0, \end{aligned} \tag{8.3}$$

then (8.2) and (8.3) imply

$$W - V \geq \beta_0 > 0, \quad x \in \mathbb{R} \times \mathbb{T}^{n-1}. \tag{8.4}$$

Set

$$\mathcal{M}(Y_{m,\ell}(v_0, w_0)) = \{u \in Y_{m,\ell}(v_0, w_0) \mid J_1(u) = b_{m,\ell}\}.$$

Then for any $j \in \mathbb{Z}$,

$$\tau_{-j}^1 V \in \mathcal{M}(Y_{m-(j,j,j,j),\ell}(v_0, w_0))$$

and also

$$\tau_{-j}^1 W \in \mathcal{M}(Y_{\bar{m}-(j,j,j,j),\bar{\ell}}(w_0, v_0)).$$

Consequently, by replacing V or W by such a phase shift, it can be assumed that (8.3) (i) is satisfied. If $j \in \mathbb{N}$, τ_j^1 shifts $\bar{m}_1$ to $\bar{m}_1 + j$, so by (8.3) (i),

$$\tau_j^1 W(x) - V(x) \geq \beta_0 > 0, \ \ x \in \mathbb{R} \times \mathbb{T}^{n-1}. \tag{8.5}$$

Similarly, for any large $p \in \mathbb{N}$, say $p \geq p^*$,

$$\tau_{-p}^1 W(x) - V(x) \geq \beta_0 > 0, \quad x \in \mathbb{R} \times \mathbb{T}^{n-1}. \tag{8.6}$$

Now a k-transition solution of (PDE) can be constructed for any $k \in \mathbb{N}$, $k \geq 2$. For even k, the solutions are homoclinic (to v_0 or w_0), while for odd k, they are heteroclinics. Choose $p = (p_0, \ldots, p_k) \in \mathbb{Z}^{k+1}$ so that $p_{i+1} > p_i$, $0 \leq i < k$.

Set $H = (h_0, \dots, h_k)$. To obtain solutions asymptotic to v_0 as $x_1 \to -\infty$, take $h_i = \tau^1_{-p_i}\varphi_i$, where $\varphi_i = W$ for even i and $\varphi_i = V$ for odd i. For solutions asymptotic to w_0 as $x_1 \to -\infty$, take $h_i = \tau^1_{-p_i}\varphi_i$, where now $\varphi_i = V$ for even i and $\varphi_i = W$ for odd i. Further assume

$$p_{i+1} - p_i \geq \nu_1 \equiv \bar{m}_1 - m_4 - \ell - \bar{\ell} - 2\nu_0 \tag{8.7}$$

and

$$p_{i+1} - p_i \geq p^*. \tag{8.8}$$

Note that $\nu_1 > 0$ via (8.3) (i). Set $\bar{h} = (\bar{h}_i)_{i\in\mathbb{Z}}$, where

$$\bar{h}_i = \begin{cases} \tau^1_{-i} h_0, & i < 0, \\ h_i, & 0 \leq i \leq k, \\ \tau^1_{i-k} h_k, & i > k. \end{cases}$$

Therefore by the above remarks, for any i and j such that $\varphi_i = W$ and $\varphi_j = V$, $\bar{h}_i - \bar{h}_j \geq \beta_0$. Hence if $\Phi_k(x) \equiv \inf\{h_i(x) \mid \varphi_i(x) = W(x)\}$ and $\Psi_k(x) \equiv \sup\{h_j(x) \mid \varphi_j(x) = V(x)\}$,

$$\Phi_k(x) - \Psi_k(x) \geq \beta_0. \tag{8.9}$$

Note also that Φ_k and Ψ_k are continuous.

Now a class of admissible functions to find k-transition solutions can be introduced. Define

$$\mathcal{Y}(\Psi_k, \Phi_k) = \{u \in W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1}) \mid \Psi_k \leq u \leq \Phi_k\}.$$

By (8.9), $\mathcal{Y}(\Psi_k, \Phi_k) \neq \emptyset$. Suppose that $u \in \mathcal{Y}(\Psi_k, \Phi_k)$ is a solution of (PDE) with $J_1(u) < \infty$. Then in a familiar fashion, $\|u - \chi\|_{L^2(T_i)} \to 0$ as $i \to -\infty$ with $\chi \in \{v_0, w_0\}$. Since $\Psi_k \leq u \leq \Phi_k$, if, e.g., $h_0 = \tau^1_{-p_0} W$, $\|u - w_0\|_{L^2(T_i)} \to 0$ as $i \to -\infty$ is not possible, i.e., $\|u - v_0\|_{L^2(T_i)} \to 0$ as $i \to -\infty$. Similarly, $h_0 = \tau^1_{-p_0} V$ implies $\|u - w_0\|_{L^2(T_i)} \to 0$ as $i \to -\infty$. Thus the asymptotic behavior of u as $x_1 \to -\infty$ is determined by h_0, and likewise as $x_1 \to \infty$, it is determined by h_p.

Let

$$c(\mathcal{Y}(\Psi_k, \Phi_k)) = \inf_{u \in \mathcal{Y}(\Psi_k, \Phi_k)} J_1(u). \tag{8.10}$$

Theorem 8.11. *Let F satisfy (F_1)–(F_2) and assume that $(*)_0$, and $(*)_1$ (for $\mathcal{M}_1(v_0, w_0)$ and $\mathcal{M}_1(w_0, v_0)$) hold. Then for each $k \in \mathbb{N}$, $k > 2$, $p \in \mathbb{Z}^{k+1}$ satisfying (8.7) and corresponding Ψ_k, Φ_k, there is a $U \in \mathcal{Y}(\Psi_k, \Phi_k)$ such that $J_1(U) = c(\mathcal{Y}(\Psi_k, \Phi_k))$. Moreover, any such minimizer is a solution of (PDE), satisfies the asymptotics associated with $\mathcal{Y}(\Psi_k, \Phi_k)$, and $\Psi_k < U < \Phi_k$.*

Proof. By Proposition 2.8, J_1 is bounded from below on $\mathcal{Y}(\Psi_k, \Phi_k)$ and therefore $c(\mathcal{Y}(\Psi_k, \Phi_k)) > -\infty$. For x_1 near $-\infty$, either (a) Ψ_k is a phase shift of V or (b) Φ_k is a phase shift of W. The same alternatives prevail for x_1 near ∞. Hence choosing $u \in \mathcal{Y}(\Psi_k, \Phi_k)$ so that $u = \Psi_k$ (resp. $u = \Phi_k$) for x_1 near $-\infty$ if (a) occurs (resp. if (b) occurs) with analogous choices for x_1 near ∞ shows that $J_1(u) < \infty$. Thus $c(\mathcal{Y}(\Psi_k, \Phi_k)) < \infty$.

By the arguments of Chapters 2–3, a minimizing sequence for (8.10) converges in $W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ to some $U \in \mathcal{Y}(\Psi_k, \Phi_k)$ with $J_1(U) < \infty$. To show that U satisfies (PDE), a local minimization property is required for members of $\mathcal{M}(Y_{m,\ell})$.

Proposition 8.12. *Any $V \in \mathcal{M}(Y_{m,\ell}(v_0, w_0))$ (resp. $W \in \mathcal{M}(Y_{\bar{m},\bar{\ell}}(w_0, v_0))$) possesses the minimization property: For any $z \in \mathbb{R} \times \mathbb{T}^{n-1}$ and small $r > 0$, V minimizes*

$$I_{r,z}(u) = \int_{B_r(z)} L(u)dx$$

over

$$E_{r,z} \equiv \{u \in W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1}) \mid u = V \ \text{ on } (\mathbb{R} \times \mathbb{T}^{n-1}) \setminus B_r(z)\}.$$

Proof. Let $z \in \mathbb{R} \times \mathbb{T}^{n-1}$ and suppose r satisfies $B_r(z) \subset \mathbb{R} \times \mathbb{T}^{n-1}$. Since $E_{r,z}$ is closed and convex and $I_{r,z}$ is weakly lower semicontinuous, there exists $\bar{u} \in E_{r,z}$ such that

$$I_{r,z}(\bar{u}) = \inf_{u \in E_{r,z}} I_{r,z}(u) \equiv \alpha_{r,z}. \tag{8.13}$$

Standard elliptic regularity arguments imply that any minimizer of $I_{r,z}$ over $E_{r,z}$ is a classical solution of (PDE) in $B_r(z)$. Moreover, as in the proof of Theorem 1.6 or Proposition 2.2,

$$\mathcal{M}(E_{r,z}) \equiv \{u \in E_{r,z} \mid I_{r,z}(u) = \alpha_{r,z}\}$$

is an ordered set. Hence it has a least element $\underline{u}$, i.e., $\underline{u}(x) \leq u(x)$ for all $x \in B_r(z)$ and $u \in \mathcal{M}(E_{r,z})$.

If

$$I_{r,z}(V) = \alpha_{r,z}, \tag{8.14}$$

we are through. Otherwise,

$$I_{r,z}(V) > \alpha_{r,z}. \tag{8.15}$$

We will show that (8.15) is not possible. It can be assumed that

$$v_0 < \underline{u} < w_0. \tag{8.16}$$

Indeed, since $V \in \mathcal{M}(Y_{m,\ell}(v_0, w_0))$, $v_0 < V < w_0$ and (8.16) is true for $x \notin B_r(z)$. If $\underline{u}(x_0) < v_0(x_0)$ for some $x_0 \in B_r(z)$,

$$\begin{aligned} I_{r,z}(\underline{u}) &= \int_{B_r(z) \cap \{\underline{u} < v_0\}} L(\underline{u})dx + \int_{B_r(z) \cap \{\underline{u} \geq v_0\}} L(\underline{u})dx \\ &> \int_{B_r(z) \cap \{\underline{u} < v_0\}} L(v_0)dx + \int_{B_r(z) \cap \{\underline{u} \geq v_0\}} L(\underline{u})dx = I_{r,z}(\max(\underline{u}, v_0)), \end{aligned} \tag{8.17}$$

since v_0 is monotone. But $\max(\underline{u}, v_0) \in E_{r,z}$, so $\max(\underline{u}, v_0) \in \mathcal{M}(E_{r,z})$. Since $\mathcal{M}(E_{r,z})$ is ordered, $\max(\underline{u}, v_0) > \underline{u}$. But then $\max(\underline{u}, v_0) = v_0$ and $v_0 \notin E_{r,z}$, a contradiction. Hence by a similar argument with w_0, for $x \in B_r(z)$, $v_0 \le \underline{u} \le w_0$. Again our usual maximum principle argument shows that equality is not possible, so (8.16) holds.

If z is not in a constraint region and r is small enough, $B_r(z)$ also avoids the constraint regions. Hence by (8.16), $\underline{u} \in Y_{m,\ell}(v_0, w_0)$ and therefore by (8.15),

$$J_1(V) > J_1(\underline{u}), \tag{8.18}$$

contrary to the minimality of V for J_1 on $Y_{m,\ell}(v_0, w_0)$. Thus (8.15) cannot hold and Proposition 8.12 is valid for such z.

Next suppose z lies in a constraint region. For the constraint regions T_i of (6.5) (i), set

$$r_1^2 = \min\{\rho_1^2 - \|V - v_0\|^2_{L^2(T_i)} \mid m_1 - \ell \le i \le m_1 - 1\}. \tag{8.19}$$

Similarly let r_2, r_3, r_4 be the analogues of r_1 for the constraint regions of (6.5) (ii)–(iv) and set

$$r_0^2 = \min_{1 \le i \le 4} r_i^2. \tag{8.20}$$

Since V satisfies the constraints with strict inequality, $r_0 > 0$. Choose r so small that

$$\|w_0 - v_0\|^2_{L^\infty(\mathbb{R}\times\mathbb{T}^{n-1})} |B_r(0)| < \frac{1}{4} r_0^2, \tag{8.21}$$

where $|B_r(0)|$ denotes the measure of $B_r(0)$. We claim that for r satisfying (8.21), $\underline{u} \in Y_{m,\ell}(v_0, w_0)$ and (8.15)–(8.18) again yield a contradiction. To see that $\underline{u}$ satisfies the constraints (6.5), suppose (6.5) (i) fails. Then for some i in $[m_1 - \ell, m_1 - 1] \cap \mathbb{Z}$,

$$\begin{aligned}\int_{T_i} (V - v_0)^2 dx < \rho_1^2 &< \int_{T_i} (\underline{u} - v_0)^2 dx \\ &= \int_{T_i \cap B_r(z)} (\underline{u} - v_0)^2 dx + \int_{T_i \setminus B_r(z)} (V - v_0)^2 dx,\end{aligned}$$

or

$$\begin{aligned}0 < r_0^2 \le \rho_1^2 - \int_{T_i} (V - v_0)^2 dx &< \int_{T_i \cap B_r(z)} \left[(\underline{u} - v_0)^2 - (V - v_0)^2\right] dx \\ &\le 2\|w_0 - v_0\|^2_{L^\infty(\mathbb{R}\times\mathbb{T}^{n-1})} |B_r(z)|\end{aligned} \tag{8.22}$$

which is contrary to (8.21). Thus for all cases (8.14) holds and V has a local minimization property.

Corollary 8.23. $\mathcal{M}(E_{r,z}) = \{V\}$.

Proof. It suffices to show that $V = \underline{u}$. The proof of Proposition 8.12 shows that $\underline{u} \in Y_{m,\ell}(v_0, w_0)$. Therefore since $I_{r,z}(\underline{u}) = I_{r,z}(V)$ and $\underline{u} = V$ in $(\mathbb{R}\times\mathbb{T}^{n-1})\setminus B_r(z)$, $J_1(\underline{u}) = J_1(V)$. Hence $\underline{u}$ is a solution of (PDE) via Theorem 6.8. But then $V-\underline{u} \geq 0$, equals 0 in $(\mathbb{R} \times \mathbb{T}^{n-1}) \setminus B_r(z)$, and satisfies a linear elliptic PDE to which the maximum principle applies. Consequently, $V - \underline{u} \equiv 0$.

Completion of Proof of Theorem 8.11. To show that U, the limit of the minimizing sequence (u_k) of (8.10), is a solution of (PDE), let $z \in \mathbb{R} \times \mathbb{T}^{n-1}$ and let $r = r(z)$ be given by Proposition 8.12. Set

$$H_{r,z} \equiv \{u \in W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1}) \mid u = U \ \text{ on } (\mathbb{R} \times \mathbb{T}^{n-1}) \setminus B_r(z)\}.$$

With $I_{r,z}$ as in Proposition 8.12, minimize $I_{r,z}$ over $H_{r,z}$. As in Proposition 8.12, there is a minimizer $\hat{u} \in H_{r,z}$ to this problem and any such minimizer is a solution of (PDE) in $B_r(z)$. Thus to prove that U is a solution of (PDE), it suffices to show that $I_{r,z}(U) = I_{r,z}(\hat{u})$.

Since $\Psi_k \leq U \leq \Phi_k$, we claim that the local minimization property of Proposition 8.12 implies

$$\Psi_k \leq \hat{u} \leq \Phi_k. \tag{8.24}$$

To verify (8.24), suppose it is false and, e.g., for some $\hat{x} \in B_r(z)$, $\hat{u}(\hat{x}) < \Psi_k(\hat{x})$. Now $\Psi_k(\hat{x}) = \tau^1_{-q}V(\hat{x})$ for some $q \in \mathbb{Z}$. Set $\varphi = \min(\tau^1_{-q}V, \hat{u})$. Replacing V by $\tau^1_{-q}V$ and m by $m + (q, q, q, q)$ in Proposition 8.12 shows that $\varphi \in E_{r,z}$. Therefore

$$I_{r,z}(\varphi) \geq I_{r,z}(\tau^1_{-q}V). \tag{8.25}$$

If there were equality in (8.25), by Corollary 8.23 $\varphi \equiv \tau^1_{-q}V$ on $\mathbb{R} \times \mathbb{T}^{n-1}$. Since $\varphi(\hat{x}) = \hat{u}(\hat{x}) < \Psi_k(\hat{x})$, $\varphi(\hat{x}) \neq \tau^1_{-q}V(\hat{x})$. Thus

$$I_{r,z}(\varphi) > I_{r,z}(\tau^1_{-q}V). \tag{8.26}$$

Set $\psi = \max(\tau^1_{-q}V, \hat{u})$. Then

$$I_{r,z}(\psi) + I_{r,z}(\varphi) = I_{r,z}(\tau^1_{-q}V) + I_{r,z}(\hat{u}) \tag{8.27}$$

and by (8.26)–(8.27),

$$I_{r,z}(\psi) < I_{r,z}(\hat{u}). \tag{8.28}$$

But $\psi \in H_{r,z}$, so (8.28) is contrary to the choice of $\hat{u}$. A similar contradiction obtains if $\hat{u}(\hat{x}) > \Phi_k(\hat{x})$. Thus (8.24) holds, and it implies $\hat{u} \in \mathcal{Y}(\Psi_k, \Phi_k)$. If $I_{r,z}(U) > I_{r,z}(\hat{u})$, set $s = I_{r,z}(U) - I_{r,z}(\hat{u})$ and define

$$\hat{u}_k = \begin{cases} \hat{u}, & x \in B_{2r}(z), \\ u_k, & x \in (\mathbb{R} \times \mathbb{T}^{n-1}) \setminus B_{3r}(z). \end{cases}$$

For the intermediate region $B_{3r}(z) \setminus B_{2r}(z)$, writing $x = z + t\theta$, where $\theta \in S^{n-1}$ and $t \in [2r, 3r]$,

$$\hat{u}_k(x) = (3 - t/r)U(z + t\theta) + (t/r - 2)u_k(z + t\theta).$$

Thus $\hat{u}_k \in \mathcal{Y}(\Psi, \Phi)$ and

$$I_{3r,z}(\hat{u}_k) \equiv I_{r,z}(\hat{u}) + \int_{B_{3r}(z)\setminus B_r(z)} L(u_k)dx + R_k. \tag{8.29}$$

Since $u_k \to U$ in $W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ as $k \to \infty$, $R_k \to 0$ and

$$I_{r,z}(u_k) \to I_{r,z}(U).$$

Thus for large k,

$$J_1(\hat{u}_k) = J_1(u_k) + I_{r,z}(\hat{u}) - I_{r,z}(u_k) + R_k \le J_1(u_k) - \frac{s}{2}, \tag{8.30}$$

contrary to (u_k) being a minimizing sequence for (8.10). Consequently $I_{r,z}(U) = I_{r,z}(\hat{u})$ and U is a solution of (PDE).

For any other $\overline{U} \in \mathcal{Y}(\Psi_k, \Phi_k)$ with $J_1(\overline{U}) = c(\mathcal{Y}(\Psi_k, \Phi_k))$, replacing U in $H_{r,z}$ by $\overline{U}$ again yields a minimizer $\bar{u} \in H_{r,z}$ as above. If $I_{r,z}(\overline{U}) > I_{r,z}(\bar{u})$, then by the above argument $J_1(\overline{U}) > J_1(\bar{u})$, contrary to the minimality of $\overline{U}$ in $\mathcal{Y}(\Psi_k, \Phi_k)$. Thus $I_{r,z}(\overline{U}) = I_{r,z}(\hat{u})$, and again $\overline{U}$ is a solution of (PDE).

Next, since U and likewise $\overline{U}$ are solutions of (PDE) in $\mathcal{Y}(\Psi_k, \Phi_k)$ with J_1 finite, by Proposition 6.53 and the form of Ψ_k and Φ_k, U and $\overline{U}$ have the desired asymptotic behavior as $x_1 \to \pm\infty$. That $J_1(U) = c(\mathcal{Y}(\Psi_k, \Phi_k))$ follows as in the proof of Theorem 3.2. Lastly, to see that $\Psi_k < U < \Phi_k$, suppose, e.g., that for some x, $U(x) = \Phi_k(x)$. Now $\Phi_k(x) = \tau^1_{-\ell}W(x)$ for some $\ell \in \mathbb{Z}$ and by its definition, $\Phi_k \le \tau^1_{-\ell}W$. Since U and $\tau^1_{-\ell}W$ are solutions of PDE with $U \le \tau^1_{-\ell}W$, and $U(x) = \tau^1_{-\ell}W(x)$, by the maximum principle, $U \equiv \tau^1_{-\ell}W$. This is possible only if $\Phi_k \equiv \tau^1_{-\ell}W$, which in turn can occur only if $k = 2$, contrary to our hypothesis.

Remark 8.31. The proof of Proposition 8.12 shows that the solution U of (PDE) given by Theorem 8.11 possesses a local minimization property and even a global one within $\mathcal{Y}(\Psi_k, \Phi_k)$.

The final result of this section is the existence of solutions that make an infinite number of transitions.

Theorem 8.32. *Under the hypotheses of Theorem 8.11, if $p = (p_j)_{j\in\mathbb{Z}}$ satisfies (8.7), then for each $k \in \mathbb{N}$ and corresponding Ψ_k, Φ_k, there is a solution U of (PDE) in $\mathcal{Y}(\Psi_k, \Phi_k)$ with $\Psi_k < U < \Phi_k$.*

Proof. Choose $k \in \mathbb{N}$ and set $p^*_k = (-p_k, \ldots, p_k)$. Take the corresponding $H_k = (h_{-k}, \ldots, h_k)$ and the associated Ψ_k, Φ_k. Invoke Theorem 8.11 to get a solution U_k of (PDE) in $\mathcal{Y}(\Psi_k, \Phi_k)$. The L^∞ bounds on Ψ_k and Φ_k uniform in k imply $C^{2,\alpha}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ bounds for U_k, uniform in k for any $\alpha \in (0, 1)$. Therefore there is a $U \in C^{2,\alpha}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ such that $U_k \to U$ in $C^2_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ along a subsequence. Hence U is a solution of (PDE) with

$$\Psi_k \leq U \leq \Phi_k, \tag{8.33}$$

i.e., $U \in \mathcal{Y}(\Psi_k, \Phi_k)$. Strict inequality in (8.33) follows as in the proof of Theorem 8.11.

Remark 8.34. Theorem 8.32 corresponds to case (iii) mentioned after Theorem 8.1. There are also versions of Theorem 8.32 corresponding to cases (i) and (ii).

Remark 8.35. As in Remark 8.31, the solution U of (PDE) of Theorem 8.32 possesses local and global minimality properties.

Remark 8.36. Unlike the earlier existence results for multitransition solutions such as Theorem 8.1 or Theorem 8.11, where the solutions were obtained by minimization, in Theorem 8.32, the solutions are obtained by an approximation argument. Thus there is no variational characterization of the solutions given by Theorem 8.32. A direct minimization approach to Theorem 8.32 remains an interesting open question. This question is akin to that of finding a variational characterization of Bangert's heteroclinic solutions [2]. Hence one possible approach would be to find a renormalized functional here in the spirit of the argument used to prove Theorem 3.2.

Chapter 9
Monotone 2-Transition Solutions

The second multitransition case mentioned in Chapter 6 will be studied here. Proceeding somewhat more generally, suppose $v_0, w_0, \widehat{v}_0, \widehat{w}_0 \in \mathcal{M}_0$, where $v_0 < w_0 \le \widehat{v}_0 < \widehat{w}_0$ and the pairs v_0, w_0 and $\widehat{v}_0, \widehat{w}_0$ satisfy $(*)_0$. The simplest special case is that of $\widehat{v}_0 = v_0 + 1$ and $\widehat{w}_0 = w_0 + 1$. The solutions constructed here will be monotone in x_1 in the sense of Theorem 3.2, i.e., $u < \tau_{-1}^1 u$. This allows us to work in a class of functions having this property and thereby use much less restrictive constraints than employed in Chapter 6 to get existence results.

Assume that $\mathcal{M}_0$ and $\mathcal{M}_1(v_0, w_0)$ have gaps, i.e., $(*)_0$ and $(*)_1$ for $\mathcal{M}_1(v_0, w_0)$ hold. Set

$$\mathcal{T}_0 = \left\{ \int_{T_0} h\, dx \mid h \in \mathcal{M}_1(v_0, w_0) \right\}.$$

Then $\mathcal{T}_0 \subset (\int_{T_0} v_0\, dx, \int_{T_0} w_0\, dx)$. By $(*)_1$, gap pairs in $\mathcal{M}_1(v_0, w_0)$ are mapped by $\int_{T_0} \cdot\, dx$ to members of $\mathcal{T}_0$ with the interval between them not in $\mathcal{T}_0$. Choose $s < t$ in a distinct pair of such intervals. Then

$$s, t \in \left(\int_{T_0} v_0\, dx,\ \int_{T_0} w_0\, dx \right) \backslash \mathcal{T}_0 \tag{9.1}$$

and

$$\mathcal{C}_0 \equiv \left\{ h \in \mathcal{M}_1(v_0, w_0) \mid s < \int_{T_0} h\, dx < t \right\} \ne \emptyset. \tag{9.2}$$

For later reference, note that $\mathcal{C}_0$, which is an ordered subset of $\mathcal{M}_1(v_0, w_0)$, has a smallest and a largest element.

Assuming that $(*)_1$ holds for $\mathcal{M}_1(\widehat{v}_0, \widehat{w}_0)$, replacing v_0 and w_0 in $\mathcal{T}_0$ by $\widehat{v}_0, \widehat{w}_0$ defines $\widehat{\mathcal{T}}_0$. Choosing $\widehat{s}, \widehat{t} \in \widehat{\mathcal{T}}_0$ defines $\widehat{\mathcal{C}}_0$ as in (9.2).

The goal here is to find solutions that shadow some $h_0 \in \mathcal{C}_0$ and $\widehat{h} \in \widehat{\mathcal{C}}_0$. To formulate such a result, a class of admissible functions will be introduced. Choose $m \in \mathbb{Z}^2$, $m = (m_1, m_2)$ with $m_1 + 4 < m_2$ and set

$$\widehat{Y}_m = \{u \in \widehat{\Gamma}_1(v_0, \widehat{w}_0) \mid u \le \tau_{-1}^1 u \ \text{ and } u \text{ satisfies (9.3)–(9.4)}\},$$

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where

$$s \leq \int_{T_{m_1}} \min(u, w_0) dx \leq t \tag{9.3}$$

and

$$\widehat{s} \leq \int_{T_{m_2}} \max(u, \widehat{v}_0) dx \leq \widehat{t}. \tag{9.4}$$

By Proposition 2.8, the functional J_1 is defined on $\widehat{Y}_m$. Set

$$\widehat{b}_m = \inf_{u \in \widehat{Y}_m} J_1(u). \tag{9.5}$$

Our main result here is:

Theorem 9.6. *Assume that F satisfies (F_1)–(F_2), $(*)_0$ holds with associated gap pairs v_0 and w_0 and $\widehat{v}_0$ and $\widehat{w}_0$, and $\mathcal{M}_1(v_0, w_0)$, $\mathcal{M}_1(\widehat{v}_0, \widehat{w}_0)$ satisfy $(*)_1$. Then there is a $U \in \widehat{Y}_m$ such that $J_1(U) = \widehat{b}_m$. If $m_2 \gg m_1$, any such U satisfies (PDE),*

$$\begin{cases} \|U - v_0\|_{W^{1,2}(T_i)} \to 0, & i \to -\infty, \\ \|U - \widehat{w}_0\|_{W^{1,2}(T_i)} \to 0, & i \to \infty, \end{cases} \tag{9.7}$$

and

$$v_0 < U < \tau_{-1}^1 U < \widehat{w}_0. \tag{9.8}$$

Moreover, shadowing occurs in the following sense:

Theorem 9.9. *Under the hypotheses of Theorem 9.6, given any $\rho, R > 0$, for $m_2 - m_1$ possibly still larger, there are functions $U_0 \in \mathcal{C}_0$ and $\widehat{U} \in \widehat{\mathcal{C}}_0$ such that*

$$\|U - \tau_{m_1}^1 U_0\|_{W^{1,2}(T_i)} \leq \rho \quad \textit{for } i \leq m_1 + R, \tag{9.10}$$

and

$$\|U - \tau_{m_2}^1 \widehat{U}\|_{W^{1,2}(T_i)} \leq \rho \quad \textit{for } i \geq m_2 - R. \tag{9.11}$$

Moreover, $U < w_0$ for $x_1 \leq m_1 + R + 1$ and $U > \hat{v}_0$ for $x_1 \geq m_2 - R$.

Remark 9.12. The freedom in choosing the parameters s and t shows that there are infinitely many different such heteroclinic solutions of (PDE) for given v_0, w_0 and $\widehat{v}_0, \widehat{w}_0$.

Remark 9.13. The sets $T_0, T_{m_i}, i = 1, 2$, are used in the integral constraints as a matter of convenience. For technical reasons, in Chapter 13 they will be replaced by the sets $B_{1/4}(p_0) = \{x \mid |x - p_0| < 1/4\}$, $\tau_{m_i}^1 B_{1/4}(p_0), i = 1, 2$, with p_0 the center of T_0. This will leave the results of the current section unchanged.

The proofs of Theorems 9.6 and 9.9 require two preliminaries. The first is an analogue of Proposition 6.74. With s, t as in (9.1), set

$$\widehat{\Lambda}_1 = \widehat{\Lambda}_1(v_0, w_0) = \{u \in \widehat{\Gamma}_1(v_0, w_0) \mid u \le \tau_{-1}^1 u \text{ and } u \text{ satisfies (9.14)}\},$$

where

$$\int_{T_0} u\, dx = \sigma \tag{9.14}$$

for $\sigma \in \{s, t\}$. Define

$$\widehat{d}_1(v_0, w_0) = \inf_{u \in \widehat{\Lambda}_1(v_0, w_0)} J_1(u). \tag{9.15}$$

Proposition 9.16. $\widehat{d}_1(v_0, w_0) > c_1(v_0, w_0)$.

Proof. Let (u_k) be a minimizing sequence for (9.15). Then as in Proposition 6.74, it can be assumed that (u_k) is bounded in $W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ and there is a $P \in W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ such that u_k converges to P weakly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$, strongly in $L^2_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$, and pointwise a.e. as $k \to \infty$. Therefore $P \in \widehat{\Lambda}_1$ and

$$J_1(P) \ge \widehat{d}_1. \tag{9.17}$$

Moreover, $J_1(P) < \infty$. Hence by Corollary 2.49 and $(*)_0$, $P \in \mathcal{M}_0$ or $P \in \Gamma_1(v_0, w_0)$. Since u satisfies (9.14), the first alternative is not possible. Thus $P \in \Gamma_1(v_0, w_0)$, and by Proposition 2.24, (2.26)–(2.27) hold. It is readily checked that $\widehat{\Lambda}_1$ satisfies (Y_1^1), so in fact $u_k \to P$ in $W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$.

We claim that $J_1(P) = \widehat{d}_1$. Indeed, the proof of (C) of Theorem 3.2 shows that $J_1(P) \le \widehat{d}_1$, so equality follows from (9.17). Since $P \in \Gamma_1(v_0, w_0)$,

$$J_1(P) \ge c_1. \tag{9.18}$$

If there is equality in (9.18), $P \in \mathcal{M}_1(v_0, w_0)$, so by Theorem 3.2, P is a solution of (PDE). But then (9.14) is in contradiction to (9.2). Hence $\widehat{d}_1 > c_1$.

Remark 9.19. Similarly $\widehat{d}_1(\widehat{v}_0, \widehat{w}_0) > c_1(\widehat{v}_0, \widehat{w}_0)$.

The next proposition is needed to prove the shadowing estimates (9.10)–(9.11).

Proposition 9.20. *For any $\varepsilon > 0$, there is a $\overline{\delta} = \overline{\delta}(\varepsilon) > 0$ such that whenever $u \in \Gamma_1(v_0, w_0)$ satisfies $J_1(u) \le c_1(v_0, w_0) + \overline{\delta}$, there is a $\Psi \in \mathcal{M}_1(v_0, w_0)$ with*

$$\|u - \Psi\|_{W^{1,2}(X_i)} \le \varepsilon \quad \textit{for all } \; i \in \mathbb{Z}.$$

Proof. If not, for some $\varepsilon > 0$, there is a sequence $(u_k) \subset \Gamma_1(v_0, w_0)$ such that $J_1(u_k) \to c_1(v_0, w_0)$ as $k \to \infty$ while for any $\Psi \in \mathcal{M}_1(v_0, w_0)$,

$$\|u_k - \Psi\|_{W^{1,2}(X_{p_k})} > \varepsilon \tag{9.21}$$

for some $p_k = p_k(\Psi) \in \mathbb{Z}$. Replacing u_k by $\tau^1_{-\ell_k} u_k$ if necessary, it can be assumed that

$$\int_{T_i} (u_k - v_0)dx \le \frac{1}{2}\int_{T_0} (w_0 - v_0)dx \le \int_{T_0} (u_k - v_0)dx \tag{9.22}$$

for all $i \in -\mathbb{N}$. Since (u_k) is a minimizing sequence for (3.1), by the proof of Theorem 3.2, there is a $U \in \mathcal{M}_1(v_0, w_0)$ such that $u_k \to U$ in $W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ as $k \to \infty$ and $J_1(U) = c_1(v_0, w_0)$. By (9.21) with $\Psi = U$,

$$\|u_k - U\|_{W^{1,2}(X_{p_k})} > \varepsilon. \tag{9.23}$$

Since $u_k \to U$ in $W^{1,2}(T_i)$ for all $i \in \mathbb{Z}$, (9.23) implies $|p_k| \to \infty$ as $k \to \infty$. Passing to a subsequence, it can be assumed that $p_k \to \infty$ or $p_k \to -\infty$ as $k \to \infty$. The argument being the same in either event, suppose $p_k \to \infty$ as $k \to \infty$.

Choose $\sigma > 0$. Since $U \in \Gamma_1(v_0, w_0)$, there is a $q \in \mathbb{Z}$ such that

$$\|U - w_0\|_{W^{1,2}(X_i)} \le \sigma \tag{9.24}$$

for all $i \in \mathbb{Z}$ with $i \ge q$. Thus for large k,

$$\|u_k - w_0\|_{W^{1,2}(X_q)} \le 2\sigma. \tag{9.25}$$

Let $f_k, g_k \in \Gamma_1(v_0, w_0)$, $h_k \in \Gamma_1(w_0)$, and let μ be defined as in (6.19)–(6.24), so

$$|J_1(u_k) - J_1(f_k)| \le \mu(\sigma) \tag{9.26}$$

and

$$J_1(f_k) = J_1(g_k) + J_1(h_k). \tag{9.27}$$

Consequently, since $J_1(u_k) \to c_1(v_0, w_0)$ as $k \to \infty$, for large k,

$$J_1(f_k) \le J_1(u_k) + \mu(\sigma) \le c_1(v_0, w_0) + 2\mu(\sigma) \le J_1(g_k) + 2\mu(\sigma). \tag{9.28}$$

Thus by (9.27)–(9.28),

$$J_1(h_k) \le 2\mu(\sigma). \tag{9.29}$$

Choose σ so small that

$$\mu(\sigma) \le \frac{1}{4}\beta(\varepsilon/2), \tag{9.30}$$

where β is as in Proposition 6.13. By (9.23)–(9.24),

$$\|u_k - w_0\|_{W^{1,2}(X_{p_k})} \ge \|u_k - U\|_{W^{1,2}(X_{p_k})} - \|U - w_0\|_{W^{1,2}(X_{p_k})} \ge \varepsilon - \sigma. \tag{9.31}$$

Thus for $\sigma < \varepsilon/2$,

$$\|u_k - w_0\|_{W^{1,2}(X_{p_k})} \geq \varepsilon/2. \tag{9.32}$$

But then since $h_k \in \Gamma_1(w_0)$ and $h_k = u_k$ on T_{p_k},

$$J_1(h_k) \geq \beta(\varepsilon/2), \tag{9.33}$$

contrary to (9.29)–(9.30).

Having completed these preliminaries, we are ready for the:

Proof of Theorem 9.6. Let (u_k) be a minimizing sequence for (9.5). By, e.g., the argument of Proposition 9.16, it can be assumed that u_k converges in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{T}^{n-1})$ to $U \in \widehat{Y}_m$ with $J_1(U) < \infty$. To show that U possesses the asymptotic behavior given by (9.7), observe that as in the proof of Corollary 2.49, as $\ell \to -\infty$, $\tau^1_{-\ell}U \to \varphi$, and as $\ell \to \infty$, $\tau^1_{-\ell}U \to \psi$, convergence being in $L^2(T_0)$. Moreover, $\varphi, \psi \in \mathcal{M}_0$, since $J_1(U) < \infty$. Clearly $\varphi \leq \psi$ with equality impossible via (9.3) or (9.4). If $\varphi \neq v_0$, by $(*)_0$ for the pair v_0, w_0, $\varphi \geq w_0$. Hence by (9.3)

$$\int_{T_0} w_0\, dx = \int_{T_{m_1}} \min(\varphi, w_0) dx \leq \int_{T_{m_1}} \min(U, w_0) dx \leq t, \tag{9.34}$$

contrary to (9.1). Thus $\varphi = v_0$ and similarly $\psi = \widehat{w}_0$. Moreover, by Proposition 2.24, the convergence to v_0 and $\widehat{w}_0$ is in $W^{1,2}(T_i)$, so (9.7) holds. It then follows as in the proof of (C) of Theorem 3.2 that $J_1(U) = \widehat{b}_m$.

Once it has been shown that U is a solution of (PDE), then as in earlier arguments, the maximum principle implies $v_0 < U < \tau^1_{-1}U < \widehat{w}_0$. The proof that for $m_2 \gg m_1$, any $U \in \widehat{Y}_m$ with $J_1(U) = \widehat{b}_m$ satisfies (PDE) consists of two parts. The first is to show that if $m_2 \gg m_1$, the constraints (9.3)–(9.4) hold with strict inequality. The second step employs a local minimization argument as in [7].

To begin, suppose that $w_0 < \widehat{v}_0$ and set

$$\begin{aligned}\widehat{Y} =&\{u \in \widehat{\Gamma}_1(w_0, \widehat{v}_0) \mid u \leq \tau^1_{-1}u \text{ and } u = w_0 \text{ in } T_i \text{ for } i \text{ near } -\infty;\\ &u = \widehat{v}_0 \text{ in } T_i \text{ for } i \text{ near } \infty\}\end{aligned}$$

and define

$$\widehat{c} = \inf_{u\in\widehat{Y}} J_1(u). \tag{9.35}$$

If $w_0 = \widehat{v}_0$, $\widehat{Y}$ and $\widehat{c}$ can be dispensed with in the following argument. With $\widehat{d}_1$ being given by Proposition 9.16 and Remark 9.19, let δ satisfy

$$0 < \delta < \frac{1}{3}\min\Big(\widehat{d}_1(v_0, w_0) - c_1(v_0, w_0), \widehat{d}_1(\widehat{v}_0, \widehat{w}_0) - c_1(\widehat{v}_0, \widehat{w}_0)\Big). \tag{9.36}$$

Choose $\alpha \in \mathcal{M}_1(v_0, w_0)$, $\beta \in \widehat{Y}$, and $\gamma \in \mathcal{M}_1(\widehat{v}_0, \widehat{w}_0)$ such that $\tau^1_{-m_1}\alpha \in \mathcal{C}_0$, $J_1(\beta) \le \widehat{c} + \delta$, and $\tau^1_{-m_2}\gamma \in \widehat{\mathcal{C}}_0$. Then for $m_2 \gg m_1$, there are $a, b \in \mathbb{Z}$ with $m_1 \ll a \ll b \ll m_2$ such that the function

$$A = \begin{cases} \alpha, & x_1 \le a, \\ w_0, & a+1 \le x_1 \le a+2, \\ \beta, & a+3 \le x_1 \le b-3, \\ \widehat{v}_0, & b-2 \le x_1 \le b-1, \\ \gamma, & b \le x_1, \end{cases}$$

and extended as usual to the remaining regions, satisfies

$$J_1(A) \le J_1(\alpha) + J_1(\beta) + J_1(\gamma) + \delta \le c_1(v_0, w_0) + \widehat{c} + c_1(\widehat{v}_0, \widehat{w}_0) + 2\delta. \quad (9.37)$$

By construction, $A \in \widehat{Y}_m$, so by (9.37),

$$\widehat{b}_m \le c_1(v_0, w_0) + \widehat{c} + c_1(\widehat{v}_0, \widehat{w}_0) + 2\delta. \quad (9.38)$$

Choose any $U \in \widehat{Y}_m$ such that $J_1(U) = \widehat{b}_m$. Set

$$f_1 = \min(U, w_0),$$
$$f_2 = \min(\widehat{v}_0, \max(U, w_0)),$$
$$f_3 = \max(U, \widehat{v}_0).$$

A straightforward analysis shows that $f_1 \in \Gamma_1(v_0, w_0)$, $f_2 \in \widehat{Y}$, and $f_3 \in \Gamma_1(\widehat{v}_0, \widehat{w}_0)$. Now suppose that one of the integral constraints (9.3)–(9.4) holds with equality, e.g.,

$$\sigma = \int_{T_{m_1}} f_1 \, dx \quad (9.39)$$

with $\sigma \in \{s, t\}$. To see that (9.39) is impossible, note that $\tau^1_{-m_1} f_1 \in \widehat{\Lambda}_1(v_0, w_0)$, so by Proposition 9.16,

$$J_1(f_1) \ge \widehat{d}_1(v_0, w_0). \quad (9.40)$$

But by earlier arguments,

$$\begin{aligned} \widehat{b}_m &= J_1(U) = J_1(f_1) + J_1(\max(U, w_0)) \\ &= J_1(f_1) + J_1(f_2) + J_1(f_3) \ge \widehat{d}_1(v_0, w_0) + \widehat{c} + c_1(\widehat{v}_0, \widehat{w}_0) \end{aligned} \quad (9.41)$$

which combined with (9.38) gives

$$\widehat{d}_1(v_0, w_0) - c_1(v_0, w_0) \leq 2\delta, \tag{9.42}$$

contrary to (9.36). Similarly, equality in (9.4) is not possible.

The final step in the proof of Theorem 9.6 is to verify that U is a solution of (PDE). To do so, choose $r \in \left(0, \frac{1}{2}\right)$ and let $z \in \mathbb{R} \times \mathbb{T}^{n-1}$. For $p \in \mathbb{Z}$, set $z_p = z + pe_1$. Let

$$E_p(z) = \{u \in W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1}) \mid u = U \ \text{ for } x \notin B_r(z_p)\}$$

and for $u \in E_p(z)$, set

$$I_p(u) = \int_{B_r(z_p)} L(u)dx.$$

Define

$$\gamma_p(z) = \inf_{u \in E_p(z)} I_p(u).$$

Then as in the proof of Proposition 8.12, there is an $f_p \in E_p(z)$ such that $I_p(f_p) = \gamma_p(z)$, f_p is a solution of (PDE) in $B_r(z_p)$, and

$$\mathcal{M}(E_p(z)) \equiv \{u \in E_p(z) \mid I_p(u) = \gamma_p(z)\}$$

is an ordered set. Observe that if $u \in \mathcal{M}(E_p(z))$, then $v_0 \leq u \leq \widehat{w}_0$. Indeed, if $\varphi = \max(u, v_0)$ and $\psi = \min(u, v_0)$, then $\varphi \in \Gamma_1(v_0, \widehat{w}_0)$ and $\psi \in \Gamma_1(v_0)$, so

$$J_1(\varphi) \leq J_1(\varphi) + J_1(\psi) = J_1(u) \tag{9.43}$$

with strict inequality if $\psi \not\equiv v_0$. Since $\varphi = u = U$ in $\mathbb{R} \times \mathbb{T}^{n-1} \setminus B_r(z_p)$, (9.43) implies

$$I_p(\varphi) \leq I_p(u) \tag{9.44}$$

with strict inequality if $\psi \not\equiv v_0$. But $u \in \mathcal{M}(E_p(z))$, so there is equality in (9.44). Hence $\psi \equiv v_0$ and $u \geq v_0$. Similarly $u \leq \widehat{w}_0$. Lastly, observe that $\mathcal{M}(E_p(z))$ is closed, so since it is ordered, it possesses a smallest element, f_p^*.

Define

$$G(U) = \begin{cases} f_p^*, & x \in B_r(z_p), \ p \in \mathbb{Z} \\ U, & x \in (\mathbb{R} \times \mathbb{T}^{n-1}) \setminus \bigcup_{i \in \mathbb{Z}} B_r(z_i). \end{cases} \tag{9.45}$$

We claim that $G(U) \in \widehat{Y}_m$. Assuming this for the moment, then

$$J_1(U) = \widehat{b}_m \leq J_1(G(U)), \tag{9.46}$$

which implies

$$I_p(U) \le I_p(f_p^*), \quad p \in \mathbb{Z}. \tag{9.47}$$

Since $U \in E_p(z)$ for all $p \in \mathbb{Z}$, (9.47) shows that $U \in \mathcal{M}(E_p(z))$ and therefore U is a solution of (PDE) in $\bigcup_{p\in\mathbb{Z}} B_r(z_p)$ for all $z \in \mathbb{R}\times\mathbb{T}^{n-1}$.

To prove that $G(U) \in \widehat{Y}_m$, by an above observation, $v_0 \le G(U) \le \widehat{w}_0$. Thus it need only be shown that $G(U)$ satisfies (9.3)–(9.4) and $\tau_{-1}^1 G(U) \ge G(U)$. Recall that U satisfies (9.3)–(9.4) with strict inequality. Moreover,

$$\int_{T_{m_1}} \min(G(U), w_0)dx = \int_{T_{m_1}} \min(U, w_0)dx \\ - \int_{\left(\bigcup_{p\in\mathbb{Z}} B_r(z_p)\right)\cap T_{m_1}} (\min(U, w_0) - \min(G(U), w_0))dx \tag{9.48}$$

and

$$\left| \int_{\left(\bigcup_{p\in\mathbb{Z}} B_r(z_p)\right)\cap T_{m_1}} ((\min(U, w_0) - \min(G(U), w_0))dx \right| \\ \le |B_r(z_0)|\|w_0 - v_0\|_{L^\infty(\mathbb{T}^n)} \le |B_r(0)|. \tag{9.49}$$

Hence for r small, $G(U)$ satisfies (9.3) and similarly (9.4).

Finally, to verify that $\tau_{-1}^1 G(U) \ge G(U)$, by the definition of G and properties of U, this reduces to checking the result for $x \in \bigcup_{p\in\mathbb{Z}} B_r(z_p)$, i.e., to showing that

$$f_{p+1}^*(x + e_1) \ge f_p^*(x), \tag{9.50}$$

for $p \in \mathbb{Z}$ and $x \in B_r(z_p)$. If (9.50) fails for some p, there is a $\xi \in B_r(z_p)$ such that

$$f_p^*(\xi) > f_{p+1}^*(\xi + e_1). \tag{9.51}$$

For $x \in B_{\frac{1}{2}}(z_p)$, set $\varphi(x) = \max(f_p^*(x), \tau_{-1}^1 f_{p+1}^*(x))$ and $\psi(x) = \min(f_p^*(x), \tau_{-1}^1 f_{p+1}^*(x))$. Therefore

$$I_p(\varphi) + I_p(\psi) = I_p(f_p^*) + I_p(\tau_{-1}^1 f_{p+1}^*) \\ = I_p(f_p^*) + I_{p+1}(f_{p+1}^*) = \gamma_p(z) + \gamma_{p+1}(z). \tag{9.52}$$

Note that for $x \in B_{\frac{1}{2}}(z_p) \setminus B_r(z_p)$, $\psi = f_p^* = U \le \tau_{-1}^1 U = \tau_{-1}^1 f_{p+1}^* = \varphi$. Extending ψ as U and φ as $\tau_{-1}^1 U$ to $(\mathbb{R}\times\mathbb{T}^{n-1}) \setminus B_{\frac{1}{2}}(z_p)$ shows that so extended, $\psi \in E_p(z)$ and $\tau_1^1\varphi \in E_{p+1}(z)$. Thus

$$I_p(\psi) \ge \gamma_p(z),\ I_p(\varphi) = I_{p+1}(\tau_1^1\varphi) \ge \gamma_{p+1}(z). \tag{9.53}$$

Comparing (9.52) and (9.53) shows that $\psi \in \mathcal{M}(E_p(z))$ and $\tau_1^1\varphi \in \mathcal{M}(E_{p+1}(z))$. By the choice of f_p^*, $f_p^* \le \psi$ in $B_r(z_p)$. Consequently, by the definition of ψ,

$$f_p^*(\xi) \le \psi(\xi) \le \tau_{-1}^1 f_{p+1}^*(\xi) = f_{p+1}^*(\xi + e_1), \tag{9.54}$$

contrary to (9.51). Thus $G(U) \in \widehat{Y}_m$, and Theorem 9.6 is proved.

Remark 9.55. (i) The proof of Theorem 9.6 shows that any $U \in \mathcal{M}_{1,m}$ has a local minimality property, since $U \in \mathcal{M}(E_p(z))$ for all small r and all $z \in \mathbb{R} \times \mathbb{T}^{n-1}$.

(ii) There is an interesting difference between the minimization values $\widehat{b}_p$ given by (9.5) and their close relatives $b_{m,\ell}$ of (6.7), at least when F is even in x_1 and we are in the simplest geometrical setting. To illustrate, suppose $\mathcal{M}_0 = \{v_0 + k | k \in \mathbb{Z}\}$, so $w_0 = v_0 + 1$ and $\mathcal{M}_1 = \{\tau_k^1 v_1 | k \in \mathbb{Z}\}$. Take $\widehat{v}_0 = w_0$, so $\widehat{w}_0 = v_0 + 2$. Then for any $p \in \mathbb{Z}^2$ and $(m, \ell) \in \mathbb{Z}^4 \times \mathbb{N}$ as in Theorems 9.6 and 6.8,

$$\widehat{b}_p > c_1(v_0, w_0) + c_1(w_0, v_0) = 2c_1(v_0, w_0) > b_{m,\ell}. \tag{9.56}$$

To see this, note first that if $\widehat{U}$ is a monotone two-transition solution of (PDE) as given by Theorem 9.6,

$$\widehat{b}_p = J_1(\widehat{U}) = J_1(\min(\widehat{U}, w_0)) + J_1(\max(\widehat{U}, w_0)) \tag{9.57}$$

with

$$\min(\widehat{U}, w_0) \in \Gamma_1(v_0, w_0) \backslash \mathcal{M}_1(v_0, w_0)$$

and

$$\max(\widehat{U}, w_0) \in \Gamma_1(\widehat{v}_0, \widehat{w}_0) \backslash \mathcal{M}_1(\widehat{v}_0, \widehat{w}_0).$$

Hence

$$\widehat{b}_p > c_1(v_0, w_0) + c_1(w_0, v_0) = 2c_1(v_0, w_0), \tag{9.58}$$

since F is even in x_1.

On the other hand, for any $u \in Y_{m,\ell}(v_0, w_0)$,

$$b_{m,\ell} \le J_1(u) = \sum_{i=-\infty}^{-1} J_{1,i}(u) + \sum_{i=0}^{\infty} J_{1,i}(u). \tag{9.59}$$

For convenience, suppose $m_3 = -m_2$. Let $v \in \mathcal{M}_1(v_0, w_0)$, so $v(-x_1, x_2, \dots, x_n) \equiv v^*(x) \in \mathcal{M}_1(w_0, v_0)$. We can assume that m, ℓ, and v are such that v satisfies (6.5) (i)–(ii) and v^* satisfies (6.5) (iii)–(iv). Therefore if

$$u(x) = \begin{cases} v(x), & x_1 \le 0 \\ v^*(x), & x_1 \ge 0, \end{cases}$$

then $u \in Y_{m,\ell}(v_0, w_0)$. Since F is even in x_1, by Remark 2.85, $J_{1,i}(u) \geq 0$ for all $i \in \mathbb{Z}$. Thus by (9.59).

$$b_{m,\ell} < J_1(v) + J_1(v^*) = 2c_1(v_0, w_0), \tag{9.60}$$

so combining (9.58) and (9.60) yields (9.56).

(iii) For the setting of the pendulum example of Remark 7.38, where v_0 corresponds to $-\pi$, $w_0 = \widehat{v}_0$ to π, and $\widehat{w}_0$ to 3π, the 2-transition solution here represents the 1-monotone motion of a pendulum that starts at $-\pi$ at $t = -\infty$, approaches and remains near π for a long time interval depending on $m_2 - m_1$, and then tends to 3π as $t \to \infty$.

Next we give the:

Proof of Theorem 9.9. The shadowing estimates (9.10)–(9.11) must be verified. Their proofs being the same, the details will be carried out for (9.10). Let $\sigma > 0$ and free for the moment. It can be assumed that δ of (9.36) further satisfies

$$2\delta < \overline{\delta}(\sigma), \tag{9.61}$$

where $\overline{\delta}$ is given by Proposition 9.20. Arguing as in (9.41), for any $U \in \widehat{Y}_m$ such that $J_1(U) = \widehat{b}_m$,

$$J_1(U) = J_1(f_1) + J_1(f_2) + J_1(f_3) \geq J_1(f_1) + \widehat{c} + c_1(\widehat{v}_0, \widehat{w}_0), \tag{9.62}$$

so by (9.38) and (9.61),

$$J_1(f_1) \leq c_1(v_0, w_0) + 2\delta < c_1(v_0, w_0) + \overline{\delta}(\sigma). \tag{9.63}$$

Hence by Proposition 9.20, there is a $\Psi \in \mathcal{M}_1(v_0, w_0)$ such that

$$\| f_1 - \Psi\|_{W^{1,2}(X_i)} \leq \sigma, \quad i \in \mathbb{Z}. \tag{9.64}$$

Note that whenever $U < w_0$ on T_i, $f_1 = U$ and (9.64) implies

$$\|U - \tau^1_{m_1} \Psi_0\|_{W^{1,2}(T_i)} \leq \sigma \tag{9.65}$$

where $\Psi_0 \equiv \tau^1_{-m_1} \Psi$. Thus to prove (9.10), it suffices to show that (A) $U < w_0$ on T_i for all $i \leq m_1 + R$ and (B) $\Psi_0 \in \mathcal{C}_0$. Toward this end, by the definition of $\mathcal{C}_0$, there are both a smallest $\underline{h}$ and largest $\overline{h}$ in $\mathcal{M}_1(v_0, w_0)$ such that

$$\overline{s} \equiv \int_{T_0} \overline{h}\, dx < s < t < \int_{T_0} \underline{h}\, dx \equiv \underline{t}. \tag{9.66}$$

We further require that

$$\sigma < \min(\underline{t} - t, s - \overline{s}). \tag{9.67}$$

Assuming (A), suppose $\Psi_0 \geq \underline{h}$. Then by (9.3), (9.64), and (9.67),

$$\underline{t} \leq \int_{T_0} \Psi_0 \, dx = \int_{T_{m_1}} \Psi \, dx \leq \int_{T_{m_1}} f_1 \, dx + \int_{T_{m_1}} |\Psi - f_1| dx \leq t + \sigma < \underline{t}, \tag{9.68}$$

which is impossible.

Similarly if $\Psi_0 \leq \overline{h}$,

$$\overline{s} \geq \int_{T_0} \Psi_0 \, dx \geq \int_{T_{m_1}} f_1 \, dx - \int_{T_{m_1}} |\Psi - f_1| dx \geq s - \sigma > \overline{s}. \tag{9.69}$$

Therefore

$$\overline{h} < \Psi_0 < \underline{h} \; (< w_0), \tag{9.70}$$

i.e., $\Psi_0 \in \mathcal{C}_0$.

It remains to verify (A):

$$U < w_0, \tag{9.71}$$

for $x \in T_i$ and $i \leq m_1 + R$. Set

$$\theta = \frac{1}{4} \min_{R+1 \leq x_1 \leq R+2} (w_0 - \underline{h}),$$

so by the monotonicity of $\underline{h}$ and (9.70),

$$0 < 4\theta \leq w_0 - \underline{h} \leq w_0 - \Psi_0, \tag{9.72}$$

for $x_1 \leq R + 2$. Define

$$\varphi = \max(f_1 - \Psi, 0).$$

Since $f_1 - \Psi = \min(U - \Psi, w_0 - \Psi)$,

$$\begin{cases} \varphi = 0 & \text{on } \{U \leq \Psi\}, \\ \varphi \geq 4\theta & \text{on } \{U \geq w_0\} \cap \{x_1 \leq m_1 + R + 2\}, \end{cases} \tag{9.73}$$

via (9.72). If (9.71) fails for some $i \leq m_1 + R$, by (9.73), there is a $\xi \in T_i$ such that $\varphi(\xi) \geq 4\theta$ and by (9.64),

$$\theta^2 \operatorname{meas}(\{\varphi \geq \theta\} \cap Z_i) \leq \int_{Z_i} \varphi^2 \, dx \leq \sigma^2. \tag{9.74}$$

Since both U and Ψ are solutions of (PDE) lying between v_0 and $\widehat{w}_0$ on $\mathbb{R} \times \mathbb{T}^{n-1}$, they are bounded in $L^\infty(\mathbb{R} \times \mathbb{T}^{n-1})$. Thus the Schauder estimates imply that there is an $M > 0$ such that

$$\|\nabla U\|_{L^\infty(\mathbb{R} \times \mathbb{T}^{n-1})}, \|\nabla \Psi\|_{L^\infty(\mathbb{R} \times \mathbb{T}^{n-1})} \leq M. \tag{9.75}$$

Choose

$$r = \min\left(\frac{1}{3}, \frac{\theta}{2M}\right). \tag{9.76}$$

Then $B_r(\xi) \subset Z_i$. Further assume that

$$\sigma < \theta\left(\frac{|B_r(0)|}{2}\right)^{1/2}. \tag{9.77}$$

Suppose $B_r(\xi) \subset (\{\varphi \geq \theta\} \cap Z_i)$. Then by (9.74),

$$|B_r(0)| = |B_r(\xi)| \leq \text{meas}(\{\varphi \geq \theta\} \cap Z_i) \leq \sigma^2/\theta^2, \tag{9.78}$$

which is contrary to (9.77). Therefore there is a $q_1 \in B_r(\xi)$ such that $\varphi(q_1) < \theta$. Choose q_2, q_3 on the line segment joining ξ and q_1 such that $\varphi(q_2) = \theta, \varphi(q_3) = 3\theta$, and $\theta \leq \varphi \leq 3\theta$ on $\ell = \{tq_2 + (1-t)q_3 \mid t \in [0,1]\}$. Then on ℓ, (9.73) shows that $\varphi = U - \Psi$. Moreover, for some $q \in \ell$,

$$\frac{2\theta}{r} \leq \frac{2\theta}{|q_3 - q_2|} = \frac{\varphi(q_3) - \varphi(q_2)}{|q_3 - q_2|} = |\nabla\varphi(q)| \leq |\nabla U(q)| + |\nabla\Psi(q)| \leq 2M, \tag{9.79}$$

contrary to (9.76). Thus (9.71) holds. A similar argument gives (9.11) and that $U > \hat{v}_0$ for $x_1 \geq m_2 - R$. The proof of Theorem 9.9 is complete.

Next some results that will be employed in Chapter 13 will be presented.

Define

$$\mathcal{M}_{1,m} = \{u \in \widehat{Y}_m \mid J_1(u) = \widehat{b}_m\}. \tag{9.80}$$

Proposition 9.81. *Under the hypotheses of Theorem 9.6, for $m_2 - m_1$ possibly still larger, $\mathcal{M}_{1,m}$ is ordered and contains a largest and smallest element.*

Proof. Let $u_1, u_2 \in \mathcal{M}_{1,m}$ and set $\varphi = \max(u_1, u_2)$ and $\psi = \min(u_1, u_2)$. We claim that $\varphi, \psi \in \widehat{Y}_m$. Assuming this for the moment,

$$2\widehat{b}_m \leq J_1(\varphi) + J_1(\psi) = J_1(u_1) + J_1(u_2), \tag{9.82}$$

so $\varphi, \psi \in \mathcal{M}_{1,m}$ and hence are solutions of (PDE) with $\varphi \geq \psi$. By the arguments following (2.5), either $\varphi \equiv \psi$ and $u_1 \equiv u_2$, or $\varphi > \psi$ in which case $u_1 > u_2$ or $u_2 > u_1$, and Proposition 9.81 is proved once the claim is established.

To verify that $\varphi, \psi \in \widehat{Y}_m$, it must be shown that they satisfy (9.3)–(9.4). Let g and f (resp. $\widehat{g}$ and $\widehat{f}$) denote the smallest and largest elements in $\mathcal{C}_0$ (resp. $\widehat{\mathcal{C}}_0$). Choose ρ so that

$$0 < \rho < \frac{1}{2}\min\left(\int_{T_0} g\,dx - s, t - \int_{T_0} f\,dx, \int_{T_0} \widehat{g}\,dx - \widehat{s}, \widehat{t} - \int_{T_0} \widehat{f}\,dx\right). \tag{9.83}$$

With this choice of ρ and say $R = 1$, invoke Theorem 9.9 to get $U_1, U_2 \in \mathcal{C}_0$ and $\widehat{U}_1, \widehat{U}_2 \in \widehat{\mathcal{C}}_0$, which shadow u_1, u_2 as in (9.10) and (9.11). Set $V_i = \tau^1_{m_i} U_i,\ i = 1, 2$. Without loss of generality, $V_2 = \max_{i=1,2} V_i$. To check that φ satisfies (9.3), suppose we have shown that

$$\|\min(\varphi, w_0) - V_2\|_{L^1(T_{m_1})} \le 2\rho. \tag{9.84}$$

Then since

$$s + 2\rho \le \int_{T_{m_1}} V_i\ dx \le t - 2\rho, \quad i = 1, 2, \tag{9.85}$$

(9.84)–(9.85) imply (9.3) for φ. Likewise, if

$$\|\min(\psi, w_0) - V_1\|_{L^1(T_{m_1})} \le 2\rho, \tag{9.86}$$

(9.85)–(9.86) give (9.3) for ψ.

To prove (9.84), note that since $w_0 \ge V_2$,

$$\begin{aligned}
\| \min(\varphi, w_0) - V_2\|_{L^1(T_{m_1})} &= \int_{T_{m_1} \cap \{\varphi > w_0\}} (w_0 - V_2) dx + \int_{T_{m_1} \cap \{\varphi \le w_0\}} |\varphi - V_2| dx \\
&\le \int_{T_{m_1}} |\varphi - V_2| dx \le \int_{T_{m_1} \cap \{u_2 \ge u_1\}} |u_2 - V_2| dx \\
&\quad + \int_{T_{m_1} \cap \{V_2 \ge u_1 > u_2\}} (V_2 - u_1) dx + \int_{T_{m_1} \cap \{u_2 < V_2 < u_1\}} (u_1 - V_2) dx \\
&\quad + \int_{T_{m_1} \cap \{u_1 > u_2 \ge V_2\}} (u_1 - V_2) dx \le \int_{T_{m_1} \cap \{u_2 \ge u_1\}} |u_2 - V_2| dx \\
&\quad + \int_{T_{m_1} \cap \{V_2 \ge u_1 > u_2\}} (V_2 - u_2) dx + \int_{T_{m_1} \cap \{u_2 < V_2 < u_1\}} (u_1 - V_1) dx \\
&\quad + \int_{T_{m_1} \cap \{u_1 > u_2 \ge V_2\}} (u_1 - V_1) dx \le \int_{T_{m_1}} (|u_2 - V_2| + |u_1 - V_1|) dx \le 2\rho.
\end{aligned} \tag{9.87}$$

Related reasoning gives (9.86). Likewise, similar arguments with V_2 and V_1 replaced by the larger and smaller of $\tau_{m_2} \widehat{U}_1$ and $\tau_{m_2} \widehat{U}_2$ yield (9.4) for φ, ψ and that $\mathcal{M}_{1,m}$ is ordered.

Finally, to show that $\mathcal{M}_{1,m}$ has a largest and smallest element, let $A = \{u(0) \mid u \in \mathcal{M}_{1,m}\}$. Since A is bounded, we can choose $\{u_k\} \subset \mathcal{M}_{1,m}$ such that $u_k(0) \to \sup A$. But $\{u_k\}$ is a minimizing sequence for J_1 over $\widehat{Y}_m$, so as in the proof of Theorem 9.6, $u_k \to u \in \mathcal{M}_{1,m}$. Since $u(0) = \max A$, u is the largest element of $\mathcal{M}_{1,m}$. Existence of a smallest element is established in essentially the same way, so the proof of Proposition 9.81 is complete.

Next, pointwise upper and lower bounds for elements of $\mathcal{M}_{1,m}$ will be obtained.

Proposition 9.88. *Under the hypotheses of Theorem 9.6, for $m_2 - m_1$ possibly still larger, if f is the largest element of $\mathcal{C}_0$, $\widehat{g}$ the smallest element of $\widehat{\mathcal{C}}_0$, and $u \in \mathcal{M}_{1,m}$, then*

$$\tau^1_{m_1} f < u < \tau^1_{m_2} \widehat{g}. \tag{9.89}$$

Proof. Let $\varphi = \max(u, \tau^1_{m_1} f)$ and $\psi = \min(u, \tau^1_{m_1} f)$, so $\psi \in \Gamma_1(v_0, w_0)$. We claim that $\varphi \in \widehat{Y}_m$. Assuming this for the moment,

$$\widehat{b}_m + c_1(v_0, w_0) \le J_1(\varphi) + J_1(\psi) = J_1(u) + J_1(\tau^1_{m_1} f) = \widehat{b}_m + c_1(v_0, w_0). \tag{9.90}$$

Hence $\varphi \in \mathcal{M}_{1,m}$ and $\varphi \ge \psi$. But $\varphi = u$ for large x_1, so $\varphi \equiv u > \tau^1_{m_1} f = \psi$. The second inequality in (9.89) follows in a similar manner.

To verify that $\varphi \in \widehat{Y}_m$, we must show that φ satisfies (9.3)–(9.4). Arguing as in Proposition 9.81, to obtain (9.3), it suffices to prove

$$\|\min(\varphi, w_0) - \tau^1_{m_1} f\|_{L^1(T_{m_1})} \le \rho, \tag{9.91}$$

where ρ is as in (9.83). To get (9.91), note that since $w_0 \ge \tau^1_{m_1} f \ge \tau^1_{m_1} U_0$, with U_0 given by Theorem 9.9,

$$\begin{aligned}
&\|\min(\varphi, w_0) - \tau^1_{m_1} f\|_{L^1(T_{m_1})} \\
&\quad \le \int_{T_{m_1} \cap \{\varphi > w_0\}} (w_0 - \tau^1_{m_1} f) dx + \int_{T_{m_1} \cap \{w_0 \ge \varphi\}} |\varphi - \tau^1_{m_1} f| dx \\
&\quad \le \|\varphi - \tau^1_{m_1} f\|_{L^1(T_{m_1})} = \int_{T_{m_1} \cap \{u > \tau^1_{m_1} f\}} (u - \tau^1_{m_1} f) dx \\
&\quad \le \int_{T_{m_1}} |u - \tau^1_{m_1} U_0| dx \le \rho.
\end{aligned} \tag{9.92}$$

A similar argument gives (9.4).

Remark 9.93. By Proposition 9.88, the definition of f and (9.10), for $m_2 - m_1$ sufficiently large and $i \le m_1 + R$,

$$\|u - \tau^1_{m_1} f\|_{L^2(T_i)} \le \|u - \tau^1_{m_1} U_0\|_{L^2(T_i)} \le \rho, \tag{9.94}$$

since u and f are each solutions of (PDE) that are bounded in $\mathbb{R} \times \mathbb{T}^{n-1}$, (9.94) and the L^p elliptic theory imply an estimate like (9.10) with U_0 replaced by f and likewise (9.11) with $\hat{U}$ replaced by $\hat{g}$.

The next result allows the comparison of elements in two different $\mathcal{M}_{1,m}$ classes. By two different classes $\mathcal{M}^i_{1,m}$, $i = 1, 2$, we mean that m_1 and m_2 are fixed but the corresponding sets $\hat{Y}^i_m$ differ via (9.3)–(9.4) where different parameters $s_i, t_i, \hat{s}_i, \hat{t}_i$ are allowed and therefore possibly different $\mathcal{C}^i_0$, $\hat{\mathcal{C}}^i_0$. As above let f_i be the largest element of $\mathcal{C}^i_0$ and $\widehat{g}_i$ the smallest element of $\widehat{\mathcal{C}}^i_0$, $i = 1, 2$.

Corollary 9.95. *Assume the hypotheses of Theorem 9.6, with $m_2 - m_1$ possibly still larger, $f_1 \le f_2$, $\widehat{g}_1 \le \widehat{g}_2$, and $u_i \in \mathcal{M}^i_{1,m}$, $i = 1, 2$. If $f_1 < f_2$ or $\widehat{g}_1 < \widehat{g}_2$, then*

$$u_1 < u_2. \tag{9.96}$$

If $f_1 = f_2$ and $\widehat{g}_1 = \widehat{g}_2$, then

$$\mathcal{M}^1_{1,m} = \mathcal{M}^2_{1,m}. \tag{9.97}$$

Furthermore, assume $f_2 \le \tau^1_{-1} f_1$ and $\widehat{g}_2 \le \tau^1_{-1}\widehat{g}_1$. If $f_2 < \tau^1_{-1} f_1$ or $\widehat{g}_2 < \tau^1_{-1}\widehat{g}_1$, then

$$u_2 < \tau^1_{-1} u_1. \tag{9.98}$$

If $f_2 = \tau^1_{-1} f_1$, $\widehat{g}_2 = \tau^1_{-1}\widehat{g}_1$, and if u_1 is the largest element of $\mathcal{M}^1_{1,m}$ or u_2 is the smallest element of $\mathcal{M}^2_{1,m}$, then

$$u_2 \le \tau^1_{-1} u_1. \tag{9.99}$$

Proof of Corollary 9.95. Let $\varphi = \max(u_1, u_2)$ and $\psi = \min(u_1, u_2)$. We claim that $\psi \in \widehat{Y}^1_m$ and $\varphi \in \widehat{Y}^2_m$. If so,

$$\widehat{b}_{m_1} + \widehat{b}_{m_2} \le J_1(\psi) + J_1(\varphi) = J_1(u_1) + J_1(u_2) = \widehat{b}_{m_1} + \widehat{b}_{m_2}.$$

Thus $\psi \in \mathcal{M}^1_{1,m}$, and $\varphi \in \mathcal{M}^2_{1,m}$ and by their definition, $\psi \le \varphi$. Since both ψ and φ are solutions of (PDE), again as following (2.5), either $\psi \equiv \varphi$ or $\psi < \varphi$. To see that the latter possibility obtains and in particular $u_1 < u_2$, suppose that $f_1 < f_2$. By Remark 9.93, for any $\rho > 0$ and $m_2 - m_1$ sufficiently large,

$$\|u_i - \tau^1_{m_1} f_i\|_{W^{1,2}(T_{m_1})} \le \rho, \tag{9.100}$$

$i = 1, 2$. Since u_i and f_i are solutions of (PDE), there is an $\omega(\rho)$ such that

$$\|u_i - \tau^1_{m_1} f_i\|_{L^\infty(T_{m_1})} \le \omega(\rho), \tag{9.101}$$

where $\omega(\rho) \to 0$ as $\rho \to 0$. Choose ρ so small that

$$2\omega(\rho) < \min_{T_0}(f_2 - f_1). \tag{9.102}$$

Then, by (9.101)–(9.102) and (9.89), for $x \in T_0$,

$$\tau^1_{-m_1}(u_2 - u_1) = (\tau^1_{-m_1} u_2 - f_2) + (f_2 - f_1) + (f_1 - \tau^1_{-m_1} u_1)$$
$$\geq f_2 - f_1 - 2\omega(\rho) > 0. \tag{9.103}$$

Thus $\phi = u_2 > u_1 = \psi$ on T_{m_1} and therefore on $\mathbb{R} \times \mathbb{T}^{n-1}$. A similar conclusion obtains if $f_1 \leq f_2$ and $g_1 < g_2$.

To verify that $\psi \in \widehat{Y}^1_m$ and $\varphi \in \widehat{Y}^2_m$, we must check that the appropriate versions of (9.3)–(9.4) hold. This follows from the argument of (9.83)–(9.87), since we can assume that ρ satisfies (9.83) for both the $i = 1, 2$ settings.

Next if $f_1 = f_2$ and $\hat{g}_1 = \hat{g}_2$, since ρ satisfies (9.83) for $i = 1, 2$, the argument of (9.83)–(9.87) shows that whenever $u \in \mathcal{M}^1_{1,m}$, then $u \in \mathcal{M}^2_{1,m}$ and conversely. Thus $\mathcal{M}^1_{1,m} = \mathcal{M}^2_{1,m}$.

Now consider the case that $f_2 \leq \tau^1_{-1} f_1$ and $\widehat{g}_2 \leq \tau^1_{-1}\widehat{g}_1$. We can analyze this by applying the first part of the corollary. To do so, define sets $\mathcal{C}^{1,*}_0 = \mathcal{C}^2_0$, $\mathcal{C}^{2,*}_0 = \tau^1_{-1}\mathcal{C}^1_0$, $\widehat{\mathcal{C}}^{1,*}_0 = \widehat{\mathcal{C}}^2_0$, $\widehat{\mathcal{C}}^{2,*}_0 = \tau^1_{-1}\widehat{\mathcal{C}}^1_0$. We wish to produce sets of solutions $\mathcal{M}^{1,*}_{1,m} = \mathcal{M}^2_{1,m}$ and $\mathcal{M}^{2,*}_{1,m} = \tau^1_{-1}\mathcal{M}^1_{1,m}$. This requires a careful definition of $s^*_i, t^*_i, \widehat{s}^*_i, \widehat{t}^*_i, i = 1, 2$. Let $s^*_1 = s_2, t^*_1 = t_2, \widehat{s}^*_1 = \widehat{s}_2, \widehat{t}^*_1 = \widehat{t}_2$, so $\mathcal{M}^{1,*}_{1,m} = \mathcal{M}^2_{1,m}$. The definition of $s^*_2, t^*_2, \widehat{s}^*_2, \widehat{t}^*_2$ is more delicate due to the translation used in the definitions of $\mathcal{C}^{2,*}_0, \widehat{\mathcal{C}}^{2,*}_0$.

Let $h_i \in \mathcal{M}_1(v_0, w_0),\ i = 1, \ldots, 4$, such that $h_2 = g_1, h_3 = f_1$ are respectively the smallest and largest elements of $\mathcal{C}^1_0$, and h_1, h_3 are the elements of $\mathcal{M}_1(v_0, w_0)$ with $h_1 < h_2,\ h_4 > h_3, h_1, h_2$ and h_3, h_4 being gap pairs. Now pick s^*_2 in the interval $\left(\int_{T_0} \tau^1_{-1} h_1\,dx, \int_{T_0} \tau^1_{-1} h_2\,dx\right)$, and t^*_2 in the interval $\left(\int_{T_0} \tau^1_{-1} h_3\,dx, \int_{T_0} \tau^1_{-1} h_4\,dx\right)$. Thus by construction,

$$\mathcal{C}^{2,*}_0 = \left\{ h \in \mathcal{M}_1(v_0, w_0) \mid s^*_2 < \int_{T_0} h\,dx < t^*_2 \right\} = \tau^1_{-1}\mathcal{C}^1_0,$$

and making analogous choices for $\hat{s}^*_2$ and $\hat{t}^*_2$, $\hat{\mathcal{C}}^{2,*}_0 = \tau^1_{-1}\hat{\mathcal{C}}^1_0$. Therefore using suggestive notation, $f^*_1 = f_2, f^*_2 = \tau^1_{-1} f_1, \hat{g}^*_1 = \hat{g}_2$, and $\hat{g}^*_2 = \tau^1_{-1}\hat{g}_1$. Hence by the first part of Corollary 9.95, $\mathcal{M}^{2,*}_{1,m} = \tau^1_{-1}\mathcal{M}^1_{1,m}$.

Finally, the last statement in Corollary 9.95 follows, since $\mathcal{M}^2_{1,m} = \tau^1_{-1}\mathcal{M}^1_{1,m}$.

We now consider further estimates required in Chapter 13. As before, assume $v_0, w_0, \widehat{v}_0, \widehat{w}_0 \in \mathcal{M}_0$, where $v_0 < w_0 \leq \widehat{v}_0 < \widehat{w}_0$ and the pairs v_0, w_0 and $\widehat{v}_0, \widehat{w}_0$ satisfy $(*)_0$. In addition, assume that there exist $v_1, w_1 \in \mathcal{M}_1(v_0, w_0)$, and $\widehat{v}_1, \widehat{w}_1 \in \mathcal{M}_1(\widehat{v}_0, \widehat{w}_0)$ where $v_1 < w_1 < \widehat{v}_1 < \widehat{w}_1$ and the pairs v_1, w_1 and $\widehat{v}_1, \widehat{w}_1$ satisfy $(*)_1$.

Define $\mathcal{C}^i_0, \widehat{\mathcal{C}}^i_0, \mathcal{M}^i_{1,m},\ i = 1, 2$, as before Corollary 9.95, choosing s_i, t_i such that $v_1 \in \mathcal{C}^1_0, w_1 \notin \mathcal{C}^1_0, v_1 \notin \mathcal{C}^2_0, w_1 \in \mathcal{C}^2_0$, i.e.,

$$t_1, s_2 \in \left(\int_{T_0} v_1\,dx, \int_{T_0} w_1\,dx \right), \tag{9.104}$$

and $\widehat{s}_i, \widehat{t}_i$ such that $\widehat{v}_1 \in \widehat{\mathcal{C}}_0^1$, $\widehat{w}_1 \notin \widehat{\mathcal{C}}_0^1$, $\widehat{v}_1 \notin \widehat{\mathcal{C}}_0^2$, $\widehat{w}_1 \in \widehat{\mathcal{C}}_0^2$, i.e.

$$\widehat{t}_1, \widehat{s}_2 \in \left(\int_{T_0} \widehat{v}_1 \, dx, \int_{T_0} \widehat{w}_1 \, dx \right). \tag{9.105}$$

Assume that $m_2 - m_1$ is large enough that Proposition 9.81 applies, so we can take U_1 to be the largest element of $\mathcal{M}_{1,m}^1$ and U_2 to be the smallest element of $\mathcal{M}_{1,m}^2$. By Corollary 9.95, $U_1 < U_2$, since $f_1 = v_1 < w_1 \leq f_2$. In addition, since there are gaps between $\tau_{-1}^1 v_1, \tau_{-1}^1 w_1$ and $\tau_1^1 \widehat{v}_1, \tau_1^1 \widehat{w}_1$, take $t_2, \widehat{s}_1$ such that $f_2 \leq \tau_{-1}^1 v_1 = \tau_{-1}^1 f_1$ and $\tau_1^1 \widehat{g}_2 = \tau_1^1 \widehat{w}_1 \leq \widehat{g}_1$, i.e., $\widehat{g}_2 \leq \tau_{-1}^1 \widehat{g}_1$, so by Corollary 9.95

$$U_1 < U_2 \leq \tau_{-1}^1 U_1. \tag{9.106}$$

Proposition 9.107. *Given U_i, $i = 1, 2$, as above, and $\sigma > 0$, there are functions $M_0(\sigma)$, $R_0(\sigma)$, and $\kappa_i(\sigma)$, with $\kappa_i(\sigma) \to 0$ as $\sigma \to 0$, $i = 1, \ldots, 5$, such that if $m_2 - m_1 \geq M_0(\sigma)$ and $R \geq R_0(\sigma)$,*

$$\|U_i - v_0\|_{W^{1,2}(T_{m_1 - R - j})} \leq \sigma, \|U_i - \hat{w}_0\|_{W^{1,2}(T_{m_2 + R + j})} \leq \sigma, \quad i = 1, 2, \tag{9.108}$$

for $j = 0, 1, 2, \ldots,$

$$|J_{1;-\infty, m_1 - R}(U_i)|, |J_{1;m_2 + R, \infty}(U_i)| \leq \kappa_1(\sigma), \quad i = 1, 2, \tag{9.109}$$

and

$$\|U_1 - U_2\|_{W^{1,2}(((-\infty, m_1 - R] \cup [m_2 + R, \infty)) \times \mathbb{T}^{n-1})} \leq \kappa_2(\sigma). \tag{9.110}$$

Suppose in addition that

$$w_0 = \widehat{v}_0 \ and \ w_1, \widehat{v}_1 \ are \ isolated \ elements \ of \ \mathcal{M}_1(v_0, w_0), \mathcal{M}_1(\widehat{v}_0, \widehat{w}_0) \tag{9.111}$$

respectively, and $t_2, \widehat{s}_1$ are chosen such that

$$\mathcal{C}_0^2 = \{w_1\}, \widehat{\mathcal{C}}_0^1 = \{\widehat{v}_1\}. \tag{9.112}$$

Define $U_3 = \max(U_1, \min(U_2, w_0))$. Then

$$\|U_i - w_0\|_{W^{1,2}(T_j)} \leq \kappa_3(\sigma), \ i = 1, 2, 3, \ j = m_1 + R, \ldots, m_2 - R, \tag{9.113}$$

$$|J_{1;m_1 + R, m_2 - R}(U_i)| \leq \kappa_4(\sigma), \ i = 1, 2, 3, \tag{9.114}$$

and

$$\|U_i - U_j\|_{W^{1,2}([m_1 + R, m_2 - R] \times \mathbb{T}^{n-1})} \leq \kappa_5(\sigma), \ i, j = 1, 2, 3. \tag{9.115}$$

Proof. Compactness properties of $\mathcal{C}_0^1$ imply that (2.26) holds uniformly in $\mathcal{C}_0^1$, so the $i = 1$ case of (9.108) follows for all large R from Theorem 9.9. The rest of (9.108) follows similarly. Define

$$H_1 = \begin{cases} v_0, & x_1 \le m_1 - R, \\ U_1, & m_1 - R + 1 \le x_1, \end{cases}$$

with the usual interpolation, noting that $H_1 \in \widehat{Y}^1_m$, so $J_1(U_i) \le J_1(H_1)$ and consequently $J_{1;-\infty,m_1-R}(U_i) \le J_{1,m_1-R}(H_1)$.

Hence by (9.108), $J_{1;-\infty,m_1-R}(U_1) \le \kappa_1(\sigma)$, where $\kappa_1(\sigma) \to 0$ as $\sigma \to 0$. To complete the proof of this case, i.e., $|J_{1;-\infty,m_1-R}(U_1)| \le \kappa_1(\sigma)$, define

$$H_2 = \begin{cases} v_0, & p - \frac{3}{2} \le x_1 \le p - 1, \\ U_1, & p \le x_1 \le m_1 - R + 1, \\ v_0, & m_1 - R + 2 \le x_1 \le m_1 - R + \frac{5}{2}, \end{cases}$$

with the usual interpolations, extended as an $(m_1 - R - p + 4)$-periodic function in x_1. Proposition 2.2 implies $0 \le J_{1;p-2,m_1-R+1}(H_2)$. Thus

$$0 \le J_{1,p-1}(H_2) + J_{1;p,m_1-R}(U_1) + J_{1,m_1-R+1}(H_2)$$

and $|J_{1,i}(H_2)| \le \kappa_1(\sigma)$ for $i = p-1, m_1-R+1$. Letting $p \to -\infty$ thus completes (9.109) for U_1. The rest of (9.109) follows similarly.

Since $v_0 \le U_i \le \hat{\omega}_0$, $i = 1, 2$, it follows as in (9.75) that there is a constant M_3 independent of m such that $|\nabla U_i| \le M_3$. Using this bound, the argument of (2.9)–(2.14) can be altered to establish

$$\left| J_{1;p,q}(u) - J_{1;p,q}(U_i) - \frac{1}{2}\|\nabla(u - U_i)\|^2_{L^2(S_{0;p,q})} \right| \le M_4\left(\int_{S_{0;p,q}} |u - U_i|\, dx + \int_{\partial S_{0;p,q}} |u - U_i| dH^{n-1} \right) \tag{9.116}$$

for $S_{0;p,q} := S_0 \cap \{p \le x_1 \le q + 1\}$. Apply (9.116) with $u = U_2$, $i = 1$, $q = m_1 - R - 1$ and let $p \to -\infty$, yielding

$$\left| J_{1;-\infty,m_1-R-1}(U_2) - J_{1;-\infty,m_1-R-1}(U_1) - \frac{1}{2}\|\nabla(U_2 - U_1)\|^2_{L^2((-\infty,m_1-R]\times\mathbb{T}^{n-1}))} \right| \le M_4\left[\int_{S_{0;-\infty,m_1-R-1}} |U_2 - U_1| dx + \int_{\{m_1-R\}\times\mathbb{T}^{n-1}} |U_2 - U_1| dH^{n-1} \right] \tag{9.117}$$

Recall that $v_0 \le U_1 \le U_2 \le \tau^1_{-1} U_1$, so

$$\int_{-\infty}^{m_1-R} (U_2 - U_1)dx_1 \le \int_{-\infty}^{m_1-R} (\tau^1_{-1}U_1 - U_1)dx_1 = \int_{m_1-R}^{m_1-R+1} (U_1 - v_0)dx_1. \tag{9.118}$$

Likewise,

$$\int_{\{m_1-R\}\times\mathbb{T}^{n-1}} (U_2 - U_1)dH^{n-1} \le \int_{\{m_1-R\}\times\mathbb{T}^{n-1}} (U_2 - v_0)dH^{n-1}. \tag{9.119}$$

By (9.108), U_i and v_0 are close in $W^{1,2}(T_{m_1-R})$, and $U_i - v_0$ are bounded in, e.g., $C^2(\mathbb{R}\times\mathbb{T}^{n-1})$ independently of m. Therefore by interpolation there is a $\kappa_6(\sigma) \to 0$ as $\sigma \to 0$ such that $\|U_i - v_0\|_{L^\infty(T_{m_1-R})} \le \kappa_6(\sigma)$, $i = 1, 2$. Consequently, for R large enough, by (9.109), (9.117)–(9.119),

$$\|\nabla(U_2 - U_1)\|^2_{L^2((-\infty,m_1-R]\times\mathbb{T}^{n-1})} \le 4\kappa_1(\sigma) + 4M_4\kappa_6(\sigma). \tag{9.120}$$

Since

$$\begin{aligned}&\|U_2 - U_1\|^2_{L^2((-\infty,m_1-R]\times\mathbb{T}^{n-1})}\\ &\quad \le \|U_2 - U_1\|_{L^\infty((-\infty,m_1-R)\times\mathbb{T}^{n-1})} \int_{S_{0;-\infty,m_1-R-1}} (U_2 - U_1)dx,\end{aligned} \tag{9.121}$$

combining (9.120), (9.118), and (9.121) with a similar estimate for the region $[m_2 + R, \infty) \times \mathbb{T}^{n-1}$ yields (9.110).

Now assume that (9.111)–(9.112) hold. Note that (9.113) is established for $i = 1$, $j = m_1 + R$, and $i = 2$, $j = m_2 - R$ in the same manner as (9.108). Therefore, since

$$\begin{aligned}-|U_1 - w_0| &\le U_1 - w_0 \le \tau^1_k U_1 - w_0 \le \tau^1_k U_\ell - w_0\\ &\le \tau^1_k U_2 - w_0 \le \tau^1_{m_2-m_1-2R} U_2 - w_0 \le |\tau^1_{m_2-m_1-2R} U_2 - w_0|\end{aligned} \tag{9.122}$$

on T_{m_1+R} for $k = 0, 1, \ldots, m_2 - m_1 - 2R$, $\ell = 1, 2, 3$, (9.113) holds with L^2 replacing $W^{1,2}$ and $\kappa_7(\sigma)$ replacing $\kappa_3(\sigma)$. Consequently, (9.113) follows as in (4.68)–(4.71).

It remains to prove (9.114)–(9.115). Let $F_i = \min(U_i, w_0)$, $G_i = \max(U_i, w_0)$, $i = 1, 2$, and define $J_1^R(u) = \sum_{i=m_1+R}^{m_2-R-1} J_{1,i}(u)$. Note that

$$J_1^R(F_i) + J_1^R(G_i) = J_1^R(U_i) + J_1^R(w_0) = J_1^R(U_i). \tag{9.123}$$

For $i = 1, 2$ define

$$V_i = \begin{cases} F_i, & x_1 \le m_2 - R, \\ U_i, & m_2 - R + 1 \le x_1, \end{cases}$$

with the usual interpolation for $m_2 - R \le x_1 \le m_2 - R + 1$. We claim that

$$J_1(U_i) \le J_1(V_i) \le J_{1;-\infty,m_1+R-1}(U_i) + J_1^R(F_i) + J_{1;m_2-R+1,\infty}(U_i) + \kappa_8(\sigma) \tag{9.124}$$

for $R \ge R_0(\sigma)$ and $m_2 - m_1 \ge M_0(R)$. The first inequality in (9.124) follows since $V_i \in \widehat{Y}^i_m$. From Theorem 9.9 and (9.111)–(9.112) we have $U_1 < w_0$ for $x_1 \le m_1 + R + 1$, so $V_i = F_i = U_i$ for such x_1, and $U_1 > w_0$ for $x_1 \ge m_2 - R$ if $m_2 - m_1 \ge M_0(R)$, so $F_i = w_0$ for $x_1 \ge m_2 - R$. Thus

$$J_1(V_i) = J_{1;-\infty,m_1+R-1}(U_i) + J_1^R(F_i) + J_{1,m_2-R}(V_i) + J_{1;m_2-R+1,\infty}(U_i).$$

Since V_i is obtained by linear interpolation between w_0 and U_i in T_{m_2-R}, by (9.113),

$$\|V_i - w_0\|_{W^{1,2}(T_{m_2-R})} \le \kappa_9(\sigma). \tag{9.125}$$

The arguments that gave (2.14) show that

$$\left| J_{1,m_2-R}(V_i) - \frac{1}{2}\|\nabla(V_i - w_0)\|^2_{L^2(T_{m_2-R})} \right| \le M_3 \int_{T_{m_2-R}} (V_i - w_0)dx, \tag{9.126}$$

so (9.125)–(9.126) yield (9.124). Now by (9.124),

$$J_1^R(U_i) \le J_1^R(F_i) + \kappa_8(\sigma), \tag{9.127}$$

so (9.123) and (9.127) show that

$$J_1^R(G_i) \le \kappa_8(\sigma). \tag{9.128}$$

Next we claim that

$$J_1^R(F_i) \le \kappa_{10}(\sigma). \tag{9.129}$$

Indeed, define

$$V_i^* = \begin{cases} U_i, & x_i \le m_1 + R, \\ G_i, & x_i \ge m_1 + R + 1, \end{cases}$$

interpolating as usual for $m_1 + R \le x_1 \le m_1 + R + 1$. Then $V_i^* \in \hat{Y}_m^1$, and as for (9.124)–(9.126),

$$\begin{aligned} J_1(U_i) \le J_1(V_i^*) &= J_{1;-\infty,m_1+R-1}(V_i^*) \\ &\quad + J_1^R(V_i^*) + J_{1;m_2-R,\infty}(V_i^*) \\ &= J_{1;-\infty,m_1+R-1}(U_i) + J_1^R(G_i) - J_{1,m_1+R}(G_i) \\ &\quad + J_{1,m_1+R}(V_i^*) + J_{1;m_2-R,\infty}(U_i) \\ &\le J_{1;-\infty,m_1+R-1}(U_i) + J_1^R(G_i) + J_{1;m_2-R,\infty}(U_i) + \kappa_{10},(\sigma) \end{aligned} \tag{9.130}$$

since $G_i = w_0$ on T_{m_1+R}, $J_{1,m_1+R}(G_i) = 0$, and

$$|J_{1,m_1+R}(V_i^*)| \le \kappa_{10}(\sigma). \tag{9.131}$$

Hence via (9.130),

$$J_1^R(U_i) \le J_1^R(G_i) + \kappa_{10}(\sigma), \tag{9.132}$$

and (9.129) follows by (9.123). Due to (9.128) and (9.132),

$$J_1^R(U_i) \le \kappa_8(\sigma) + \kappa_{10}(\sigma) \equiv \kappa_{11}(\sigma). \tag{9.133}$$

Note that

$$J_1^R(U_3) + J_1^R(F_1) = J_1^R(U_1) + J_1^R(F_2), \tag{9.134}$$

since $\min(U_1, F_2) = F_1$, and by definition $U_3 = \max(U_1, F_2)$. Therefore (9.123), (9.128)–(9.129) and (9.134) imply

$$J_1^R(U_3) = J_1^R(G_1) + J_1^R(F_2) \le \kappa_{12}(\sigma). \tag{9.135}$$

Now to complete the proof of (9.114), we need lower bounds for $J_1^R(U_i)$. For $i = 1, 2, 3$, let

$$W_i = \begin{cases} w_0, & m_1 + R - 2 \le x_1 \le m_1 + R, \\ U_i, & m_1 + R + 1 \le x_1 \le m_2 - R, \\ w_0, & m_2 - R + 1 \le x_1 \le m_2 - R + 2, \end{cases}$$

with the usual interpolations, extended as an $(m_2 - m_1 - 2R + 4)$- periodic function in x_1. Then by Proposition 2.2,

$$0 \le J_1^{R-2}(W_i) = J_1^R(U_i) - J_{1,m_1+R}(U_i) + J_{1,m_1+R}(W_i) + J_{1,m_2-R}(W_i) \tag{9.136}$$

and

$$|J_{1,m_1+R}(U_i)|, |J_{1,m_1+R}(W_i)|, |J_{1,m_2-R}(W_i)| \leq \kappa_{13}(\sigma) \tag{9.137}$$

via earlier estimates. Hence

$$-3\kappa_{13}(\sigma) \leq J_1^R(U_i) \tag{9.138}$$

and (9.133), (9.135), and (9.138) yield (9.114).

Finally, to prove (9.115), first apply (9.116) with $p = m_1 + R$, $q = m_2 - R$, $u = U_j$, $j = 1, 2, 3$, and $i = 1, 2$ in conjunction with (9.137) and (9.114) to get

$$\begin{aligned} &\frac{1}{2}\|\nabla(U_j - U_i)\|^2_{L^2(S_0^R)} \\ &\quad \leq M_4 \left(\int_{S_0^R} |U_j - U_i| \, dx + \int_{\partial S_0^R} |U_j - U_i| \, dH^{n-1} \right) + \kappa_{14}(\sigma) \end{aligned} \tag{9.139}$$

for $S_0^R = S_{0;m_1+R,m_2-R}$. Recall that $U_1 \leq U_3 \leq U_2 \leq \tau_{-1}^1 U_1$ and that

$$\int_{m_1+R}^{m_2-R+1} (\tau_{-1}^1 U_1 - U_1) \, dx_1 = \int_{m_2-R+1}^{m_2-R+2} (U_1 - w_0) \, dx_1 - \int_{m_1+R}^{m_1+R+1} (U_1 - w_0) \, dx_1 \tag{9.140}$$

is uniformly small for $m_2 - m_1 \geq M_2(R)$, and R large due to (9.113), so (9.139) implies

$$\|\nabla(U_j - U_i)\|_{L^2(S_0^R)} \leq \kappa_{15}(\sigma), \quad i = 1, 2, \quad j = 1, 2, 3, \tag{9.141}$$

for $R \geq R_2(\sigma)$, $m_2 - m_1 \geq M_3(R(\sigma))$. Thus arguing as for (9.110) gives (9.115).

Chapter 10
Monotone Multitransition Solutions

Having established the existence of monotone 2-transition solutions of (PDE), now in the spirit of Chapter 8, we can ask for monotone k-transition solutions or even infinite-transition solutions. For the latter case, in contrast to Chapter 8, where all transitions take place in a single gap, infinitely many gaps are involved. This makes for several possibilities ranging from degenerate settings such as that in which a bounded sequence of distinct gap pairs φ_i, ψ_i having smaller and smaller gaps converges to some $\varphi \in \mathcal{M}_0$ to the generic case of $\mathcal{M}_0 = \{v + j \,|\, j \in \mathbb{Z}\}$ where $v \in \mathcal{M}_0$. We confine our study to multitransition solutions generated by a finite number of gap pairs, say $\varphi_i < \psi_i$, $1 \leq i \leq p$ (where $\psi_i \leq \varphi_{i+1}$ and $\psi_p \leq \varphi_1 + 1$), together with their additive counterparts $(\varphi_i + j, \psi_i + j)$ for $j \in \mathbb{Z}$. This contains the generic case as well as the setting of Theorem 9.6. It will also enable us to find associated infinite-transition solutions.

Thus let

$$\mathcal{S} = \mathcal{S}(\varphi_1, \psi_1, \ldots, \varphi_p, \psi_p) = \{\varphi_i + j, \psi_i + j \mid 1 \leq i \leq p, j \in \mathbb{Z}\}.$$

Since $\mathcal{S}$ is an ordered subset of $\mathcal{M}_0$, it can also be expressed as

$$\mathcal{S} = \{\hat{v}_i < \hat{w}_i \mid i \in \mathbb{Z}\},$$

where $\hat{v}_{i+p} = \hat{v}_i + 1$, $\hat{w}_{i+p} = \hat{w}_i + 1$. It is notationally more convenient to use this second formulation of $\mathcal{S}$.

Choosing k consecutive gap pairs in $\mathcal{S}$, say $\hat{v}_i < \hat{w}_i$, $1 \leq i \leq k$, and assuming that $(*)_1$ holds for each pair, Theorem 9.6 can be extended to cover this setting. However, the most direct extension of the earlier proof leads to versions of (9.38) and (9.42) with 2δ replaced by ξ_k with $\xi_k \to \infty$ as $k \to \infty$. Thus (9.36) must be strengthened to an estimate of the form

$$0 < \xi_k \delta < \min_{1 \leq i \leq k} \Big(\hat{d}_1(\hat{v}_i, \hat{w}_i) - c_1(\hat{v}_i, \hat{w}_i) \Big), \tag{10.1}$$

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and (10.1) in turn implies that as $k \to \infty$, all of the differences $m_{i+1} - m_i$ become infinite. This information is too weak to use to obtain monotone infinite-transition solutions of (PDE) as limits of the finite-transition case. Thus next we prove a k-transition result that gives better lower bounds for the differences $m_{i+1} - m_i$ and permits us to treat the infinite-transition case.

Toward this end, for $i \in \mathbb{Z}$, define

$$\mathcal{T}_i = \left\{ \int_{T_0} h \, dx \mid h \in \mathcal{M}_1(\hat{v}_i, \hat{w}_i) \right\}.$$

By $(*)_1$ for $\hat{v}_i$, $\hat{w}_i$, the set of real numbers $\mathcal{T}_i$ has infinitely many gaps. For $1 \le i \le p$, as in Chapter 9, choose $s_i < t_i$ lying in a distinct pair of such gaps, so

$$s_i, t_i \in \left(\int_{T_0} \hat{v}_i \, dx, \int_{T_0} \hat{w}_i \, dx \right) \setminus \mathcal{T}_i. \tag{10.2}$$

Since $\hat{v}_{j+p} = \hat{v}_j + 1$ and $\hat{w}_{j+p} = \hat{w}_j + 1$ for all $j \in \mathbb{Z}$, we can take $s_{i+lp} = s_i + l$ and $t_{i+lp} = t_i + l$ for all $l \in \mathbb{Z}$ and $1 \le i \le p$.

For future cutting and pasting arguments, it will be necessary to approximate the members of $\mathcal{M}_1(\hat{v}_i, \hat{w}_i)$. Let

$$\delta_i = \hat{d}_1(\hat{v}_i, \hat{w}_i) - c_1(\hat{v}_i, \hat{w}_i). \tag{10.3}$$

Note that $c_1(\hat{v}_{i+p}, \hat{w}_{i+p}) = c_1(\hat{v}_i, \hat{w}_i)$ and $\hat{d}_1(\hat{v}_{i+p}, \hat{w}_{i+p}) = \hat{d}_1(\hat{v}_i, \hat{w}_i)$. Therefore $\delta_{i+p} = \delta_i$ for all $i \in \mathbb{Z}$. Set

$$\delta = \min_{1 \le i \le p} \delta_i. \tag{10.4}$$

Proposition 10.5. *Let $m_q \in \mathbb{Z}$,*

$$M \ge \max_{1 \le i \le p} (c_1(\hat{v}_i, \hat{w}_i) + 1), \tag{10.6}$$

and

$$0 < \sigma < \min_{1 \le i \le p} \left(\int_{T_0} \hat{w}_i \, dx - t_i, s_i - \int_{T_0} \hat{v}_i \, dx \right). \tag{10.7}$$

Then there are a $\widehat{\Phi}_q \in \Gamma_1(\hat{v}_q, \hat{w}_q)$ and $\overline{\ell} = \overline{\ell}(\sigma, M) \in \mathbb{N}$ such that

$$s_q < \int_{T_{m_q}} \widehat{\Phi}_q dx < t_q, \tag{10.8}$$

$\widehat{\Phi}_q = \hat{v}_q$ for $x_1 < m_q - \overline{\ell} - 1$; $\widehat{\Phi}_q = \hat{w}_q$ for $x_1 > m_q + \overline{\ell} + 1$, and $\hat{\Phi}_q \le \tau_{-1}^1 \hat{\Phi}_q$. Moreover, for any $\hat{\delta} > 0$ and $\sigma = \sigma(\hat{\delta})$ sufficiently small,

$$J_1(\widehat{\Phi}_q) \le c_1(\hat{v}_q, \hat{w}_q) + \hat{\delta}/7. \tag{10.9}$$

Proof. Let $\Phi_q \in \mathcal{M}_1(\hat{v}_q, \hat{w}_q)$ satisfy (10.8). Since $J_1(\Phi_q) \le M$, by Proposition 6.27, there are an $\ell_q \in \mathbb{N}$ depending on σ, M and $\hat{v}_q, \hat{w}_q$, an $i \in [m_q - \ell_q + 2, m_q - 2)$, and a $\varphi \in \{\hat{v}_q, \hat{w}_q\}$ such that

$$\|\Phi_q - \varphi\|_{L^2(X_i)} \le \sigma. \tag{10.10}$$

Similarly, there are a $j \in (m_q + 2, m_q + \ell_q - 2)$ and $\psi \in \{\hat{v}_q, \hat{w}_q\}$ such that

$$\|\Phi_q - \psi\|_{L^2(X_j)} \le \sigma. \tag{10.11}$$

We claim that $\varphi = \hat{v}_q$ and $\psi = \hat{w}_q$. To see this, note that if

$$\|\Phi_q - \hat{w}_q\|_{L^2(X_i)} \le \sigma, \tag{10.12}$$

then

$$\int_{T_i} (\hat{w}_q - \Phi_q) dx \le \|\hat{w}_q - \Phi_q\|_{L^2(T_i)} \le \sigma. \tag{10.13}$$

By the monotonicity of Φ_q and (10.8),

$$\int_{T_i} (\hat{w}_q - \Phi_q) dx \ge \int_{T_{m_q}} (\hat{w}_q - \Phi_q) dx \ge \int_{T_0} \hat{w}_q dx - t_q. \tag{10.14}$$

Thus for σ satisfying (10.7), (10.13), and (10.14) show that $\varphi = \hat{v}_q$. Similarly, $\psi = \hat{w}_q$.

By the monotonicity of Φ_q again,

$$\|\Phi_q - \hat{v}_q\|_{L^2(X_s)} \le \sigma, \quad s \le i.$$

Arguing as in (6.57),

$$\|\Phi_q - \hat{v}_q\|_{W^{1,2}(Z_s)} \le M_3 \sigma, \quad s \le i. \tag{10.15}$$

A priori ℓ_q depends on the gap pair $\hat{v}_q, \hat{w}_q$. But since we are dealing with $\mathcal{S}$, the finitely generated set of gap pairs, $\overline{\ell} = \sup_{q \in \mathbb{Z}} \ell_q = \max_{1 \le q \le p} \ell_q$.

Define

$$\widehat{\Phi}_q = \begin{cases} \hat{v}_q, & x_1 \le m_q - \overline{\ell} - 1, \\ \Phi_q, & m_q - \overline{\ell} \le x_1 \le m_q + \overline{\ell}, \\ \hat{w}_q, & m_q + \overline{\ell} + 1 \le x_1, \end{cases} \tag{10.16}$$

and extend $\widehat{\Phi}_q$ to the remaining regions via the usual interpolation. Thus $\widehat{\Phi}_q \le \tau_{-1}^1 \widehat{\Phi}_q$, $\widehat{\Phi}_q \in \Gamma_1(\hat{v}_q, \hat{w}_q)$, and for σ sufficiently small, by (10.15) and its analogue for $\hat{w}_q$,

$$\left|J_{1,m_q - \overline{\ell} - 1},(\widehat{\Phi}_q)\right|, \left|J_{1,m_q + \overline{\ell}}(\widehat{\Phi}_q)\right| \le \frac{\hat{\delta}}{42} \tag{10.17}$$

and

$$\begin{aligned} J_1(\widehat{\Phi}_q) =& J_1(\Phi_q) + J_{1,m_q-\overline{\ell}-1}(\widehat{\Phi}_q) \\ &+ J_{1,m_q+\overline{\ell}}(\widehat{\Phi}_q) - J_{1;-\infty,m_1-\overline{\ell}-1}(\Phi_q) - J_{1;m_q+\overline{\ell},\infty}(\Phi_q) \\ \leq& c_1(\hat{v}_q, \hat{w}_q) + \frac{\hat{\delta}}{21} + \text{tail terms}. \end{aligned} \tag{10.18}$$

The tail terms can be estimated as in the proof of Theorem 3.2. Note that

$$\|\Phi_q - \hat{w}_q\|_{W^{1,2}(T_s)} \ \text{ and } \ J_{1;s,\infty}(\Phi_q) \to 0 \tag{10.19}$$

as $s \to \infty$. For large s, set

$$f_q = \begin{cases} \Phi_q, & m_q + \overline{\ell} \leq x_1 \leq s, \\ \tau^1_{s+1-(m_q+\overline{\ell})}\Phi_q, & s+1 \leq x_1, \end{cases} \tag{10.20}$$

with the usual interpolation in the remaining intervals. Then by Proposition 2.2,

$$J_{1;m_q+\overline{\ell},s}(f_q) \geq 0,$$

so by (10.19) for large s,

$$-\sum_{m_q+\overline{\ell}}^{\infty} J_{1,i}(\Phi_q) \leq -J_{1,s}(\Phi_q) + J_{1,s}(f_q) - \sum_{s+1}^{\infty} J_{1,i}(\Phi_q) \leq \frac{\hat{\delta}}{21} - \sum_{s+1}^{\infty} J_{1,i}(\Phi_q). \tag{10.21}$$

Letting $s \to \infty$ in (10.21) and using (10.19) gives

$$\sum_{m_q+\overline{\ell}}^{\infty} J_{1,i}(\Phi_q) \leq \frac{\hat{\delta}}{21}. \tag{10.22}$$

With a similar estimate for the remaining tail term, (10.22) and (10.18) yield (10.9), and Proposition 10.5 is proved.

Next define

$$\begin{aligned} Y_q =& \{u \in \Gamma_1(\hat{w}_q, \hat{v}_{q+1}) \mid u \leq \tau^1_{-1}u, u = \hat{w}_q \ \text{ in } T_i \ \text{ for large negative } \ i, \\ & \text{and } \ u = \hat{v}_{q+1} \ \text{ in } T_i \ \text{ for large positive } \ i\}. \end{aligned}$$

Set

$$\hat{c}_q = \inf_{u \in Y_q} J_1(u). \tag{10.23}$$

By the definition of $\hat{c}_q$, there is a $\widehat{\Psi}_q \in Y_q$ such that

$$J_1(\widehat{\Psi}_q) \leq \hat{c}_q + \frac{\hat{\delta}}{7}. \tag{10.24}$$

Since $\tau_{-j}^1 \widehat{\Psi}_q \in Y_q$ for all $j \in \mathbb{Z}$, it can be assumed that there is an $\hat{\ell}_q \in \mathbb{N}$ such that $\widehat{\Psi}_q = \hat{w}_q$ for $x_1 \leq 0$ and $\widehat{\Psi}_q = \hat{v}_{q+1}$ for $x_1 \geq \hat{\ell}_q$. As for ℓ_q, we can and do replace $\hat{\ell}_q$ by $\hat{\ell} = \sup_{q \in \mathbb{Z}} \hat{\ell}_q$ with $\hat{\ell}$ independent of q.

Now a generalization of Theorem 9.6 can be formulated. Choose $k \in \mathbb{N}$ and $i \in \mathbb{Z}$. Then there are associated consecutive gap pairs $\hat{v}_i, \hat{w}_i, \ldots, \hat{v}_{i+k-1}, \hat{w}_{i+k-1}$ in $\mathcal{S}$. Choose $m \in \mathbb{Z}^\infty$, i.e., $m = (m_i)_{i \in \mathbb{Z}}$ with $m_i \in \mathbb{Z}$ and $m_i + 4 < m_{i+1}$. The class of admissible functions here is

$$\begin{aligned} \widehat{Y}_{(m_i, m_{i+k-1})} &\equiv \widehat{Y}_{(m_i, m_{i+k-1})}(\hat{v}_i, \hat{w}_{i+k-1}) \\ &= \{u \in \widehat{\Gamma}_1(\hat{v}_i, \hat{w}_{i+k-1}) | \; u \leq \tau_{-1}^1 u \;\text{ and }\; u \text{ satisfies (10.25)}, \\ &\quad\; i \leq j \leq i+k-1\}, \end{aligned}$$

where

$$s_j \leq \int_{T_{m_j}} f_j(u)\, dx \leq t_j, \quad i \leq j \leq i+k-1, \tag{10.25}$$

and the functions $f_j(u)$, $i \leq j \leq i+k-1$, are defined via

$$f_j(u) = \min(\max(u, \hat{v}_j), \hat{w}_j).$$

Note that $f_i(u) = \min(u, \hat{w}_i)$ and $f_{i+k-1}(u) = \max(u, \hat{v}_{i+k-1})$. Set

$$\hat{b}_{(m_i, m_{i+k-1})} = \inf_{u \in \widehat{Y}_{(m_i, m_{i+k-1})}} J_1(u). \tag{10.26}$$

Now the extension of Theorem 9.6 is:

Theorem 10.27. *Suppose F satisfies (F_1)–(F_2). Let $i \in \mathbb{Z}$ and $k \in \mathbb{N}$. If $(\hat{v}_j, \hat{w}_j)$ is a gap pair in $\mathcal{S}$ and $(*)_1$ holds for $\mathcal{M}_1(\hat{v}_j, \hat{w}_j)$, for all j such that $i \leq j \leq i+k-1$, then:*

1^o *There is a $U = U_{(m_i, m_{i+k-1})} \in \hat{Y}_{(m_i, m_{i+k-1})}$ such that $J_1(U) = b_{(m_i, m_{i+k-1})}$.*
2^o *There is a $\nu \in \mathbb{N}$ (independent of i and k) such that if $m_{j+1} - m_j \geq \nu$, $i \leq j \leq i+k-2$, then any such U is a solution of (PDE),*

$$\begin{cases} \|U - \hat{v}_1\|_{W^{1,2}(T_i)} \to 0, & i \to -\infty, \\ \|U - \hat{w}_k\|_{W^{1,2}(T_i)} \to 0, & i \to \infty, \end{cases} \tag{10.28}$$

$$\hat{v}_1 < U < \tau_{-1}^{1} U < \hat{w}_k. \tag{10.29}$$

Moreover, there is an $\omega > 0$ *(independent of* i *and* k*) such that*

$$\hat{b}_{(m_i, m_{i+k-1})} \leq \sum_{j=i}^{i+k-1} c_1(\hat{v}_j, \hat{w}_j) + \sum_{j=i}^{i+k-2} \hat{c}_j + k\omega \tag{10.30}$$

and

$$J_1(f_j(U)) \leq c_1(\hat{v}_j, \hat{w}_j) + \omega. \tag{10.31}$$

Remark 10.32. Theorem 9.6 follows from Theorem 10.27 by first identifying v_0, w_0, e.g., with $\hat{v}_1, \hat{w}_1$. Then $\hat{v}, \hat{w}$ corresponds to $\hat{v}_\ell, \hat{w}_\ell$ for some $\ell > 1$. Thus in the more careful bookkeeping of the current setting, we keep track of intermediate transitions that were ignored earlier.

Remark 10.33. The constants ω and ν of Theorem 10.27 depend on the parameters of the problem, which will be chosen in the course of the proof. However, for now, before proving the theorem, we will define ω and ν. Let $\bar{\delta}(\epsilon)$ be the function defined in Proposition 9.20. Then

$$\omega = \omega(\delta, \epsilon, p) = \min(\delta, \bar{\delta}(\epsilon)) \tag{10.34}$$

for an appropriately chosen ϵ. Let

$$M = \max_{q \in \mathbb{Z}} c_1(\hat{v}_q, \hat{w}_q) + 1, \tag{10.35}$$

$l_0(\theta, M)$ be as given by Proposition 6.27, $\bar{\ell} = \bar{\ell}(\sigma, M)$ as given by Proposition 10.5, and $\hat{\ell} = \hat{\ell}(\omega)$ as defined following (10.24). Then

$$\nu = 8(l_0(\theta, M) + \bar{\ell}(\sigma, M) + \hat{\ell}(\omega)) \tag{10.36}$$

for appropriate θ, σ.

Proof of Theorem 10.27. The proof is by induction on k. The case of $k = 1$ and any $i \in \mathbb{Z}$ follows from the definition of $\mathcal{C}_i$ and the fact that $f_i(U) = U$. No restrictions on ω or ν are needed.

Assume that Theorem 10.27 has been proved for $k \geq 1$ and any $i \in \mathbb{Z}$. We will show that the theorem holds for $k + 1$. By the argument of Theorem 9.6, there is a $U = U_{(m_i, m_{i+k})}$ such that $J_1(U) = \hat{b}_{(m_{i+1}, m_{i+k})}$.

We claim that there is strict inequality in (10.25) for $u = U_{(m_i, m_{i+k})}$. The arguments of Theorem 9.6 then show that $U_{(m_i, m_{i+k})}$ is a solution of (PDE).

To verify (10.25) with strict inequality we distinguish between the cases of $j = i$ and $j = i + k$, which are simpler, and $i + 1 \leq j \leq i + k - 1$. Suppose $j = i$. Set

$$\varphi_j(U) = \min(\max(U, \hat{w}_j), \hat{v}_{j+1})$$

and

$$\xi_j(U) = \min(\max(U, \hat{v}_{j+1}), \hat{w}_{i+k}).$$

Then

$$\hat{b}_{(m_i, m_{i+k})} = J_1(U) = J_1(f_i(U)) + J_1(\varphi_i(U)) + J_1(\xi_i(U)). \tag{10.37}$$

Note that $\varphi_i(U) \in Y_i$ and $\xi_i(U) \in \widehat{Y}_{(m_{i+1}, m_{i+k})}$. If (10.25) fails for $j = i$, then $f_i(U) \in \widehat{\Lambda}_1(\hat{v}_i, \hat{w}_i)$, and (10.37) implies

$$\hat{b}_{(m_i, m_{i+k})} \geq \hat{d}_1(\hat{v}_i, \hat{w}_i) + \hat{c}_i + \hat{b}_{(m_{i+1}, m_{i+k})}. \tag{10.38}$$

We claim that

$$\hat{b}_{(m_i, m_{i+k})} \leq c_1(\hat{v}_i, \hat{w}_i) + \hat{c}_i + \hat{b}_{(m_{i+1}, m_{i+k})} + \frac{\delta}{2}. \tag{10.39}$$

Assuming (10.39) for now, by (10.38)–(10.39),

$$\hat{d}_1(\hat{v}_i, \hat{w}_i) \leq c_1(\hat{v}_i, \hat{w}_i) + \frac{\delta}{2}. \tag{10.40}$$

But (10.40) is contrary to (10.3)–(10.4).

Thus (10.25) holds for $j = i$.

To verify (10.39), note first that by the inductive hypothesis, there is a $U_{(m_{i+1}, m_{i+k})} \in \widehat{Y}_{(m_{i+1}, m_{i+k})}$ such that $J_1(U_{(m_{i+1}, m_{i+k})}) = \hat{b}_{(m_{i+1}, m_{i+k})}$ and by (10.31),

$$J_1(f_{i+1}(U_{(m_{i+1}, m_{i+k})})) \leq c_1((\hat{v}_{i+1}, \hat{w}_{i+1})) + \omega. \tag{10.41}$$

By (10.34), $\omega \leq \bar{\delta}(\epsilon)$, where ϵ is free for the moment. Note that by earlier remarks, $\bar{\delta}$ of Proposition 9.20 can be assumed to be independent of i and k, but will depend on p. By Proposition 9.20, there is an $h \in \mathcal{M}_1(\hat{v}_{i+1}, \hat{w}_{i+1})$ such that

$$\| f_{i+1}(U_{(m_{i+1}, m_{i+k})}) - h \|_{W^{1,2}(X_s)} \leq \epsilon \tag{10.42}$$

for all $s \in \mathbb{Z}$.

Let $\underline{h}_q$ be the smallest member of $\mathcal{M}_1(\hat{v}_q, \hat{w}_q)$ such that

$$t_q < \int_{T_0} \underline{h}_q \, dx \equiv \underline{t}_q. \tag{10.43}$$

Similarly, let $\bar{h}_q$ be the largest member of $\mathcal{M}_1(\hat{v}_q, \hat{w}_q)$ such that

$$\bar{s}_q \equiv \int_{T_0} \bar{h}_{q+1}\, dx < s_q. \tag{10.44}$$

Note that

$$\bar{h}_{q+pl} = \bar{h}_q + l \tag{10.45}$$

for $l \in \mathbb{Z}$.

Suppose that ϵ satisfies

$$0 < \epsilon < \inf_{q\in\mathbb{Z}} \min(\underline{t}_q - t_q, s_q - \bar{s}_q) = \min_{1\le q\le p} \min(\underline{t}_q - t_q, s_q - \bar{s}_q). \tag{10.46}$$

Observe that this choice of ϵ fixes $\omega = \omega(\delta, \epsilon, p)$.

Now employing the argument of (9.64)–(9.79) with $R = 1$ and $\epsilon, \hat{v}_{i+1}, \hat{w}_{i+1}$ replacing σ, v_0, w_0, etc, we obtain

$$U_{(m_{i+1}, m_{i+k})} < \hat{w}_{i+1}, \quad x_1 \le m_{i+1} + 1, \tag{10.47}$$

and $\tau^1_{-(m_1+1)} h \in \mathcal{C}_{i+1}$. (Here we are taking into account that (10.45) implies that the argument is independent of i and k.) With M as in (10.35), by (10.31), Proposition 6.27 with σ replaced by θ, and (10.47), there are $\ell_0 = \ell_0(\theta, M) \in \mathbb{N}$ and $q \in [m_{i+1} - 2\ell_0 + 2, m_{i+1} - 2]$ such that

$$\|U_{(m_{i+1}, m_{i+k})} - \varphi\|_{L^2(X_q)} \le \theta \tag{10.48}$$

for some $\varphi \in \{\hat{v}_{i+1}, \hat{w}_{i+1}\}$. Hence if θ satisfies (10.7), the argument of (10.10)–(10.14) shows that $\varphi = \hat{v}_{i+1}$. As in (10.15),

$$\|U_{(m_{i+1}, m_{i+k})} - \hat{v}_{i+1}\|_{W^{1,2}(Z_q)} \le M_3 \theta. \tag{10.49}$$

Set

$$\overline{U} = \begin{cases} U_{(m_{i+1}, m_{i+k})}, & x_1 \notin Z_q, \\ \hat{v}_{i+1}, & x_1 \in T_q, \end{cases} \tag{10.50}$$

with the usual interpolation in $Z_q \backslash T_q$. Then as in earlier arguments,

$$|J_1(\overline{U}) - J_1(U_{(m_{i+1}, m_{i+k})})| \le \kappa(\theta) \tag{10.51}$$

with $\kappa(\theta) \to 0$ as $\theta \to 0$. Set

$$\hat{U} = \begin{cases} \hat{v}_{i+1}, & x_1 \le q + 1, \\ \overline{U}, & x_1 \ge q + 1, \end{cases} \tag{10.52}$$

and

$$U^* = \begin{cases} \overline{U}, & x_1 \leq q+1, \\ \hat{v}_{i+1}, & x_1 \geq q+1. \end{cases} \tag{10.53}$$

Then $\widehat{U} \in \widehat{Y}_{(m_{i+1}, m_{i+k})}$ and $U^* \in \Gamma_1(\hat{v}_{i+1})$, so

$$J_1(U^*) > 0. \tag{10.54}$$

Therefore by (10.50)–(10.54),

$$\begin{aligned} J_1(\widehat{U}) &\leq J_1(U^*) + J_1(\widehat{U}) = J_1(\overline{U}) \leq J_1(U_{(m_{i+1}, m_{i+k})}) + \kappa(\theta) \\ &= \hat{b}_{(m_{i+1}, m_{i+k})} + \kappa(\theta) \leq \hat{b}_{(m_{i+1}, m_{i+k})} + \frac{\omega}{6} \end{aligned} \tag{10.55}$$

provided that $\theta = \theta(\omega)$ is sufficiently small. Further choose σ so that Proposition 10.5 holds with $\hat{\delta} = \frac{\omega}{6}$ and take $\hat{l} = \hat{l}(\omega)$ as given following (10.24) with associated $\hat{\Psi}_i$ such that $J_1(\hat{\Psi}_i) \leq \hat{c}_i + \frac{\omega}{6}$. With ν as in (10.36), glue $\hat{\Phi}_i$ to $\tau^1_{\hat{l}} \hat{\Psi}_i$ to $\hat{U}$ in the natural fashion, producing $W \in \widehat{Y}_{(m_i, m_{i+k})}$ with

$$J_1(W) \leq c_1(\hat{v}_i, \hat{w}_i) + \hat{c}_i + \hat{b}_{(m_{i+1}, m_{i+k})} + \frac{\omega}{2}. \tag{10.56}$$

Consequently, by (10.34) and (10.56), (10.39) holds for $j = i$ and similarly for $j = i + k$.

For the remaining cases of $i + 1 \leq j \leq i + k - 1$, similar ideas are used, so we will be sketchy.

Set

$$\psi_j(U) = \min(\max(U, \hat{v}_i), \hat{w}_{j-1}).$$

As in (10.37),

$$\hat{b}_{(m_i, m_{i+k})} = J_1(\psi_j(U)) + J_1(\varphi_{j-1}(U)) + J_1(f_j(U)) + J_1(\varphi_j(U)) + J_1(\xi_j(U)) \tag{10.57}$$

with $\psi_j(U) \in \widehat{Y}_{(m_i, m_{j-1})}$, $\varphi_{j-1}(U) \in Y_{j-1}$, $\varphi_j(U) \in Y_j$, and $\xi_j(U) \in \widehat{Y}_{(m_{j+1}, m_{i+k})}$. If (10.25) fails for j, $f_j(U) \in \Lambda_1(\hat{v}_j, \hat{w}_j)$ and as earlier,

$$\hat{b}_{(m_i, m_{i+k})} \geq \hat{b}_{(m_i, m_{j-1})} + \hat{c}_{j-1} + \hat{d}_1(\hat{v}_j, \hat{w}_j) + \hat{c}_j + \hat{b}_{(m_{j+1}, m_{i+k})}. \tag{10.58}$$

Our earlier argument with the same choice of parameters yields an upper bound for $\hat{b}_{(m_i,m_{i+k})}$:

$$\hat{b}_{(m_i,m_{i+k})} \leq \hat{b}_{(m_i,m_{j-1})} + \hat{c}_{j-1} + c_1(\hat{v}_j, \hat{w}_j) + \hat{c}_j + \hat{b}_{(m_{j+1},m_{i+k})} + \frac{5\omega}{6}, \quad (10.59)$$

and again (10.58)–(10.59) and (10.34) are contrary to (10.3)–(10.4).

It now follows for all cases that $U_{(m_i,m_{i+k})}$ is a solution of (PDE) and it remains only to verify (10.30) and (10.31) at level $k+1$.

The upper bound (10.30) is immediate from the choice of ν and gluing $\hat{\Phi}_i, \ldots, \hat{\Phi}_{i+k}$ to appropriate shifts of $\hat{\Psi}_i, \ldots, \hat{\Psi}_{i+k-1}$. To get (10.31), note first that (10.57) implies

$$\hat{b}_{(m_i,m_{i+k})} \geq \hat{b}_{(m_i,m_{j-1})} + \hat{c}_{j-1} + J_1(f_j(U_{(m_i,m_{i+k})})) + \hat{c}_j + \hat{b}_{(m_{j+1},m_{i+k})}, \quad (10.60)$$

so by (10.59),

$$J_1(f_j(U)) \leq c_1(\hat{v}_j, \hat{w}_j) + \frac{5\omega}{6} \leq c_1(\hat{v}_j, \hat{w}_j) + \omega. \quad (10.61)$$

The proof of Theorem 10.27 is now complete.

As a quick application of Theorem 10.27, the existence of monotone infinite transition solutions of (PDE) can be established. Let $\mathcal{S}$ be as earlier with associated sets $\mathcal{T}_i$ and $s_i < t_i$ as in (10.2), $i \in \mathbb{Z}$. Let ν be as given by Theorem 10.27 and let $m \in \mathbb{Z}^\infty$ with $m_{i+1} - m_i \geq \nu(p)$. Now set

$$\begin{aligned}\widehat{Y}_m =&\{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}^{n-1}, \mathbb{R}) \mid u \leq \tau_{-1}^1 u \ \text{ and} \\ &u \ \text{ satisfies (10.62) at index } i, \quad i \in \mathbb{Z}\}\end{aligned}$$

where

$$s_i \leq \int_{T_{m_i}} f_i(u)dx \leq t_i. \quad (10.62)$$

Theorem 10.63. *Under the above hypothesis, there is a $U \in \widehat{Y}_m$ satisfying (PDE) and $U < \tau_{-1}^1 U$.*

Proof. For each $k \in \mathbb{N}$, take $\hat{m}(k) = (m_{-k}, \ldots, m_k) \in \mathbb{Z}^{2k+1}$. By Theorem 10.27, there is a solution U_k of (PDE) in $\widehat{Y}_{\hat{m}(k)}(\hat{v}_{-k}, \hat{w}_k)$. The functions U_k are bounded in $C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}^{n-1}, \mathbb{R})$ for any $\alpha \in (0,1)$. Therefore along a subsequence, U_k converges to U, a solution of (PDE) satisfying (10.62) for all $i \in \mathbb{Z}$. Therefore $U \in \widehat{Y}_m$. Moreover, as in earlier results, $U < \tau_{-1}^1 U$.

Remark 10.64. The argument of the proof of Theorem 10.63 works equally well if $m \in \mathbb{Z}^\infty$ is replaced by $m \in \mathbb{N}^\infty$ or $m \in (-\mathbb{N})^\infty$. In the process, we obtain a solution of (PDE) that is heteroclinic to, e.g., $\hat{v}_1$ as $x_1 \to -\infty$ or to $\hat{w}_1$ as $x_1 \to \infty$. For example, suppose $m \in \mathbb{N}^\infty$ with $m_1 = 0$ and $m_i = (i-1)\nu$ for $i \geq 1$. Then the corresponding solution U of (PDE) lies between the periodic functions $\hat{v}_1$ and $\hat{w}_1$ for $x_1 \leq 0$. Therefore as $x_1 \to -\infty$, U has rotation vector 0 associated with it. On the other hand, U is unbounded as $x_1 \to \infty$. More precisely, for $x_1 \geq 0$, if $U(z)$ lies between $\hat{v}_1 + j$ and $\hat{w}_1 + j$, then $U(z + k(p+1)\nu e_1)$ lies between $\hat{v}_1 + j + k$ and $\hat{w}_1 + j + k$, i.e., U has associated rotation vector $(\frac{1}{(p+1)\nu}, 0, \ldots, 0)$ as $x_1 \to \infty$.

Remark 10.65. Just as in Remark 8.36, an open question is whether one can give a direct minimization characterization of the infinite-transition solutions of Theorem 10.63.

Chapter 11
A Mixed Case

Two rather different types of multitransition solutions were studied in Chapters 6–10: those lying between a given gap pair $v_0 < w_0$ and those that have the monotonicity property $u < \tau_{-1}^1 u$, and cross gaps. The goal of this section is to combine these two cases. Thus we seek solutions of (PDE) that are heteroclinics or homoclinics as a function of x_1, lie in prescribed gaps for x_1 near $\pm\infty$, and undergo a prescribed number of transitions between a given set of gap pairs. Roughly speaking, such solutions can be obtained by concatenating those of Chapters 6–10. Different kinds of results are possible depending on how precisely one seeks to shadow the states that are glued together.

By way of illustration suppose that $v_i^* < w_i^*, i = 1, 2$, are arbitrary gap pairs with

$$v_1^* < w_1^* \leq v_2^* < w_2^*.$$

Restricting ourselves to the simplest possibilities for solutions of mixed type, there are six cases to consider: (i) heteroclinics from v_1^* (resp. v_2^*) to v_2^* (resp. v_1^*) that also shadow w_2^* over a long x_1 interval; (ii) homoclinics to v_1^* (resp.w_2^*) that also shadow w_2^* (resp. v_1^*) over a long x_1 interval; and (iii) heteroclinics from w_1^* (resp. w_2^*) to w_2^* (resp. w_1^*) that also shadow v_1^* over a long x_1 interval. The simplest result for any of these mixed cases would be to merely prove the corresponding existence statement. A more careful theorem would take account of the number of gap pairs lying between v_1^* and w_2^* and would provide a solution that shadows heteroclinics in some or all of these gaps.

To minimize technicalities but at the same time indicate how to handle the new difficulties associated with mixed cases, we first prove a result for case (i) that gives a crude version of shadowing. Then the case of k prescribed gap pairs will be discussed. Lastly, a few remarks will be made about how to treat an infinite number of gap pairs somewhat as in Chapter 10.

To formulate the main theorem that we will prove, suppose $v_1^* < w_1^*$, $v_2^* < w_2^*$ are given gap pairs in $\mathcal{M}_0$ with $w_1^* < v_2^*$. If $w_1^* = v_2^*$, some simplifications can be made in our arguments. We seek a solution U of (PDE) that is heteroclinic in x_1

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from v_1^* to v_2^* and that is close to w_2^* for a large intermediate region. The solution U is also required to be periodic in $x_2, \dots, x_n$. In the spirit of Chapter 6, choose $m \in \mathbb{Z}^3$ with $m_{i+1} > m_i$, $i = 1, 2$, and $\ell \in \mathbb{N}$. As the class of admissible functions we take

$$Y_{m,\ell}^* = \{u \in W_{\mathrm{loc}}^{1,2}(\mathbb{R} \times \mathbb{T}^{n-1}) |\ v_1^* \le u \le w_2^* \text{ and } u \text{ satisfies (11.1)–(11.3)}\},$$

where

$$\begin{cases} \text{(i) } \|u - v_1^*\|_{L^2(T_i)} \to 0, & i \to -\infty, \\ \text{(ii) } \|u - v_2^*\|_{L^2(T_i)} \to 0, & i \to \infty, \end{cases} \tag{11.1}$$

$$\begin{cases} \text{(i) } \|u - v_1^*\|_{L^2(T_i)} \le \rho_1, & m_1 - \ell \le i \le m_1 - 1, \\ \text{(ii) } \|u - w_2^*\|_{L^2(T_i)} \le \rho_2, & m_2 - \ell \le i \le m_2 + \ell - 1, \\ \text{(iii) } \|u - v_2^*\|_{L^2(T_i)} \le \rho_3, & m_3 \le i \le m_3 + \ell - 1, \end{cases} \tag{11.2}$$

and

$$\begin{cases} \text{(i) } u \le w_1^* \ m_1 - \ell \le x_1 \le m_1, \\ \text{(ii) } u \ge v_2^* \ m_2 - \ell \le x_1 \le m_2 + \ell, \\ \text{(iii) } u \ge v_2^* \ m_3 \le x_1 \le m_3 + \ell. \end{cases} \tag{11.3}$$

Note that in contrast to earlier sections, the additional pointwise constraints (11.3) are required here. The constants ρ_i are related to those of Chapter 6 and satisfy

$$\begin{cases} \text{(i) } \rho_1 \in (0, \frac{1}{2}\|w_1^* - v_1^*\|_{L^2(T_0)}) \setminus \{\|u - v_1^*\|_{L^2(T_0)} \mid \\ \qquad u \in \mathcal{M}_1(v_1^*, w_1^*) \cup \mathcal{M}_1(w_1^*, v_1^*)\}, \\ \text{(ii) } \rho_2 \in (0, \frac{1}{2}\|w_2^* - v_2^*\|_{L^2(T_0)}) \setminus \{\|u - w_2^*\|_{L^2(T_0)} \mid \\ \qquad u \in \mathcal{M}_1(v_2^*, w_2^*) \cup \mathcal{M}_1(w_2^*, v_2^*)\}, \\ \text{(iii) } \rho_3 \in (0, \frac{1}{2}\|w_2^* - v_2^*\|_{L^2(T_0)}) \setminus \{\|u - v_2^*\|_{L^2(T_0)} \mid \\ \qquad u \in \mathcal{M}_1(v_2^*, w_2^*) \cup \mathcal{M}_1(w_2^*, v_2^*)\}. \end{cases} \tag{11.4}$$

Define

$$c_{m,\ell}^* = \inf_{u \in Y_{m,\ell}^*} J_1(u). \tag{11.5}$$

Then we have:

Theorem 11.6. *Suppose* (F_1)–(F_2) *hold,* (v_1^*, w_1^*) *and* (v_2^*, w_2^*) *are gap pairs satisfying*

$$v_1^* < w_1^* \le v_2^* < w_2^*,$$

and $(*)_1$ *holds for*

$$\bigcup_{i=1}^{2}(\mathcal{M}_1(v_i^*, w_i^*) \cup \mathcal{M}_1(w_i^*, v_i^*)).$$

Then for ℓ *sufficiently large, there is a* $U \in Y_{m,\ell}^*$ *such that* $J_1(U) = c_{m,\ell}^*$. *Moreover, for* $m_{i+1} - m_i$ *sufficiently large,* $i = 1, 2$, *any such* U *is a classical solution of (PDE).*

Proof. As usual, let (u_k) be a minimizing sequence for (11.5), so there is an $M > 0$ such that $J_1(u_k) \le M$ for all $k \in \mathbb{N}$. Since $Y_{m,\ell}^*$ satisfies (Y_1^1), by earlier arguments it can be assumed that there is a $U \in W_{\mathrm{loc}}^{1,2}(\mathbb{R} \times \mathbb{T}^{n-1})$ such that $u_k \to U$ pointwise a.e. and in $W_{\mathrm{loc}}^{1,2}(\mathbb{R} \times \mathbb{T}^{n-1})$,

$$J_1(U) \le M, \tag{11.7}$$

and U satisfies (11.2)–(11.3). Moreover, with the aid of (Y_2^1) and earlier arguments, U is a solution of (PDE) in the nonconstraint regions.

Next, as in the proof of Theorem 6.8, we will show: (A) There is an X_i in each constraint region such that (PDE) is satisfied in X_i, (B) $U \in Y_{m,\ell}^*$ and $J_1(U) = c_{m,\ell}^*$, and (C) U satisfies the L^2 constraints (11.2) with strict inequality. What remains is to prove that (D) U is C^2 and satisfies (PDE) globally. This step is more difficult to carry out than in the earlier cases due to the extra pointwise constraints (11.3).

Proof of (A). To begin, let

$$\begin{aligned} f_1^*(U) &= \min(U, w_1^*), \\ f_2^*(U) &= \max(\min(U, v_2^*), w_1^*), \\ f_3^*(U) &= \max(U, v_2^*). \end{aligned}$$

Thus $f_1^*(U) \in \widehat{\Gamma}_1(v_1^*, w_1^*) \subset \widehat{\Gamma}_1(v_1^*, w_2^*)$, and likewise $f_2^*(U)$, $f_3^*(U) \in \widehat{\Gamma}_1(v_1^*, w_2^*)$ and $f_3^*(U) \in \widehat{\Gamma}_1(v_2^*, w_2^*)$. Then

$$J_1(U) = \sum_{i=1}^{3} J_1(f_i^*(U)), \tag{11.8}$$

so by (11.7)–(11.8) and Proposition 2.8, there is a $K_1 > 0$ depending on v_1^* and w_2^* such that

$$J_1(f_1^*(U)), J_1(f_3^*(U)) \le M + 2K_1. \tag{11.9}$$

Consequently, with σ free for the moment, by Proposition 6.27, if $\ell \ge \ell_0(\sigma, M + 2K_1)$, there is an $i_1 \in [m_1 - \ell + 2, m_1 - 2] \cap \mathbb{Z}$ such that

$$\| f_1^*(U) - \varphi \|_{L^2(X_{i_1})} < \sigma,$$

where $\varphi \in \{v_1^*, w_1^*\}$. Since $f_1^*(U) = U$ on $[m_1 - \ell, m_1] \times \mathbb{T}^{n-1}$ via (11.3),

$$\|U - \varphi\|_{L^2(X_{i_1})} < \sigma. \tag{11.10}$$

We claim that $\varphi = v_1^*$. Otherwise,

$$\|U - w_1^*\|_{L^2(X_{i_1})} < \sigma, \tag{11.11}$$

but by (11.2) (i),

$$\|U - v_1^*\|_{L^2(X_{i_1})}^2 \le 5\rho_1^2. \tag{11.12}$$

Thus (11.11)–(11.12) imply

$$\sqrt{5}\rho_1 \ge \|w_1^* - v_1^*\|_{L^2(X_{i_1})} - \|U - w_1^*\|_{L^2(X_{i_1})} \ge \sqrt{5}\|w_1^* - v_1^*\|_{L^2(T_0)} - \sigma. \tag{11.13}$$

Choosing σ such that

$$0 < \sigma < \frac{\sqrt{5}}{2} \min_{j=1,2} \|w_j^* - v_j^*\|_{L^2(T_0)}, \tag{11.14}$$

(11.13) shows that

$$2\rho_1 \ge \|w_1^* - v_1^*\|_{L^2(T_0)}, \tag{11.15}$$

which is contrary to (11.4). Thus

$$\|U - v_1^*\|_{L^2(X_{i_1})} < \sigma. \tag{11.16}$$

To complete the verification of (A), we will show that (11.16) implies that U is a solution of (PDE) in X_{i_1}. A variant of the proof of part (A) of Theorem 3.2 will be employed. Let $z \in T_j \subset X_{i_1}$ and let r, φ be as in (Y_2^1). Then for $|t|$ small, by (11.16),

$$\|u_k + t\varphi - v_1^*\|_{L^2(T_j)} < \rho_1 \tag{11.17}$$

for large k. Set

$$\varphi_k = \begin{cases} \max(u_k + t\varphi, v_1^*), & x_1 \le i_1 + 3, \\ u_k, & x_1 \ge i_1 + 3, \end{cases}$$

and

$$\psi_k = \begin{cases} \min(u_k + t\varphi, v_1^*), & x_1 \le i_1 + 3, \\ v_1^*, & x_1 \ge i_1 + 3. \end{cases}$$

Then $\psi_k \in \Gamma_1(v_1^*)$, and by Theorem 2.72,

$$\begin{aligned} J_{1;-\infty,i_1+2}(u_k + t\varphi) &= J_{1;-\infty,i_1+2}(\varphi_k) + J_{1;-\infty,i_1+2}(\psi_k) \\ &= J_{1;-\infty,i_1+2}(\varphi_k) + J_1(\psi_k) \\ &\ge J_{1;-\infty,i_1+2}(\varphi_k). \end{aligned} \tag{11.18}$$

Therefore

$$J_1(u_k + t\varphi) \geq J_1(\varphi_k). \tag{11.19}$$

Since (11.17) implies

$$\|\varphi_k - v_1^*\|_{L^2(T_j)} < \rho_1, \tag{11.20}$$

if $\varphi_k \leq w_1^*$ in T_j, then $\varphi_k \in Y_{m,\ell}^*$. However, this may not be the case, so one more modification of φ_k is necessary. Set

$$\chi_k = \begin{cases} \min(\varphi_k, w_1^*), & x_1 \leq i_1 + 3, \\ \varphi_k \ (= u_k), & x_1 \geq i_1 + 3, \end{cases}$$

and

$$\zeta_k = \begin{cases} \max(\varphi_k, w_1^*), & x_1 \leq i_1 + 3 \\ w_1^*, & x_1 \geq i_1 + 3. \end{cases}$$

Then $\chi_k \in Y_{m,\ell}^*$ and $\zeta_k \in \Gamma_1(w_1^*)$, so as in (11.18),

$$\begin{aligned} J_{1;-\infty,i_1+2}(\varphi_k) &= J_{1;-\infty,i_1+2}(\chi_k) + J_{1;-\infty,i_1+2}(\zeta_k) \\ &= J_{1;-\infty,i_1+2}(\chi_k) + J_1(\zeta_k) \geq J_{1;-\infty,i_1+2}(\chi_k). \end{aligned} \tag{11.21}$$

Hence

$$J_1(\varphi_k) \geq J_1(\chi_k) \geq c_{m,\ell}^*. \tag{11.22}$$

But by (11.19) and (11.22), (2.65) is satisfied, so as in the proof of Proposition 2.64, U satisfies (PDE) in X_{i_1}.

A similar argument gives sets X_{i_2} and X_{i_3} in the two other constraint regions.

Proof of (B). The first main task here is to show that U satisfies (11.1). The first step is to verify that $U \leq w_1^*$ for $x_1 \leq m_1$. For this, it suffices to prove that $u_k \leq w_1^*$ for $x_1 \leq m_1$. Arguing as in (11.21)–(11.22), set

$$\overline{u}_k = \begin{cases} \min(u_k, w_1^*), & x_1 \leq i_1, \\ u_k, & x_1 \geq i_1, \end{cases}$$

and

$$\tilde{u}_k = \begin{cases} \max(u_k, w_1^*), & x_1 \leq i_1, \\ w_1^*, & x_1 \geq i_1. \end{cases}$$

Then $\overline{u}_k \in Y_{m,\ell}^*$, and $\tilde{u}_k \in \Gamma_1(w_1^*)$ so

$$\begin{aligned} J_{1;-\infty,i_1-1}(u_k) &= J_{1;-\infty,i_1-1}(\overline{u}_k) + J_{1;-\infty,i_1-1}(\tilde{u}_k) \\ &= J_{1;-\infty,i_1-1}(\overline{u}_k) + J_1(\tilde{u}_k) \geq J_{1;-\infty,i_1-1}(\overline{u}_k) \end{aligned}$$

and

$$J_1(u_k) \geq J_1(\overline{u}_k), \tag{11.23}$$

where $\bar{u}_k \le w_1^*$ for $x_1 \le m_1$. Now (11.23) shows that $\bar{u}_k$ is also a minimizing sequence for (11.5), so we can assume that $u_k, U \le w_1^*$ for $x_1 \le m_1$.

Next we will verify (11.1) for U. Once that is shown, $U \in Y_{m,\ell}^*$ and the argument of Theorem 3.2 show that $J_1(U) = c_{m,\ell}^*$. Condition (11.1) (i) will be checked; (11.1) (ii) follows similarly.

Since $f_1^*(U) \in \widehat{\Gamma}_1(v_1^*, w_1^*)$, with $f_1^*(U) = U$ for $x_1 \le m_1$, and $f_1^*(U)$ satisfies the hypotheses of Proposition 6.53, U satisfies (11.1) (i) or

$$\|U - w_1^*\|_{L^2(T_i)} \to 0, \quad i \to -\infty. \tag{11.24}$$

To exclude (11.24), we argue as in the proof of Theorem 6.8. More precisely, if (11.24) holds, let

$$\gamma = \frac{1}{2} \min_{1 \le j \le 3} \rho_j. \tag{11.25}$$

Then for all $p \in \mathbb{N}$ near $-\infty$,

$$\|U - v_1^*\|_{L^2(T_p)} \ge 2\gamma. \tag{11.26}$$

Hence for a fixed such p and large k,

$$\|u_k - v_1^*\|_{L^2(T_p)} \ge \gamma. \tag{11.27}$$

Modify u_k in Z_{i_1} to produce a function h_k with $h_k = u_k$ for $x_1 \le i_1$ and $h_k = v_1^*$ for $x_1 \ge i_1 + 1$. Then h_k belongs to $\Gamma_1(v_1^*)$ and satisfies (11.27). Hence by Proposition 6.13,

$$J_1(h_k) \ge \beta(\gamma). \tag{11.28}$$

Define $H_k = v_1^*$ for $x_1 \le i_1$ and $H_k = u_k$ for $x_1 \ge i_1 + 1$ with the usual interpolation in T_{i_1}. Then $H_k \in Y_{m,\ell}^*$, and for $k \ge k_0(\sigma)$,

$$|J_{1,i_1}(u_k)|, |J_{1,i_1}(H_k)|, |J_{1,i_1}(h_k)| \le \kappa(\sigma), \tag{11.29}$$

where $\kappa(\sigma) \to 0$ as $\sigma \to 0$. We require that σ be so small that

$$6\kappa(\sigma) < \beta(\gamma). \tag{11.30}$$

Now by (11.28)–(11.30),

$$\begin{aligned} J_1(H_k) &= J_{1,i_1}(H_k) + J_{1,i_1+1,\infty}(u_k) \\ &= J_{1,i_1}(H_k) + J_1(u_k) - J_{1;-\infty,i_1}(u_k) \\ &= J_{1,i_1}(H_k) + J_1(u_k) - J_1(h_k) + J_{1;i_1}(h_k) - J_{1,i_1}(u_k) \\ &\le J_1(u_k) - \beta(\gamma) + 3\kappa(\sigma) \le J_1(u_k) - \frac{1}{2}\beta(\gamma). \end{aligned} \tag{11.31}$$

But (11.31) is contrary to (u_k) being a minimizing sequence. Consequently, (11.1) (i) and similarly (11.1) (ii) hold for U.

Proof of (C). Showing that U satisfies the constraints (11.2) with strict inequality involves a combination of arguments of Chapters 6–10. To begin, suppose $j \in [m_1 - \ell, m_1 - 1] \cap \mathbb{Z}$. By (11.16), the only cases of interest are (a) $j < i_1 - 2$ and (b) $j > i_1 + 2$. For j satisfying (a), if

$$\|U - v_1^*\|_{L^2(T_j)} = \rho_1, \tag{11.32}$$

then as in an earlier argument, define

$$h = \begin{cases} U, & x_1 \le i_1, \\ v_1^*, & i_1 + 1 \le x_1, \end{cases}$$

and interpolate as usual in T_{i_1} so that $h \in \Gamma_1(v_1^*)$,

$$J_1(h) \ge \beta(\rho_1), \tag{11.33}$$

and

$$|J_{1,i_1}(h)| \le \kappa(\sigma). \tag{11.34}$$

Set

$$H = \begin{cases} v_1^*, & x_1 \le i_1 - 1, \\ U, & i_1 \le x_1, \end{cases}$$

and again interpolate in $T_{i_1 - 1}$ so that $H \in Y_{m,\ell}^*$ and

$$|J_{1,i_1-1}(H)| \le \kappa(\sigma). \tag{11.35}$$

Then $J_1(H) \ge J_1(U)$, so

$$J_{1,i_1-1}(H) \ge J_{1;-\infty,i_1-1}(U). \tag{11.36}$$

Hence by (11.33)–(11.36),

$$\kappa(\sigma) \ge J_{1,i_1-1}(H) \ge J_{1;-\infty,i_1-1}(U) \ge J_1(h) - \kappa(\sigma) \ge \beta(\rho_1) - \kappa(\sigma). \tag{11.37}$$

But (11.37) is contrary to (11.30), so (11.32) is not possible for case (a).

Next suppose (b) occurs together with (11.32). Set

$$\Lambda_1(v_1^*, w_1^*) = \{u \in \Gamma_1(v_1^*, w_1^*) \,|\, \|u - v_1^*\|_{L^2(T_0)} = \rho_1\}$$

and

$$d_1(v_1^*, w_1^*) = \inf_{u \in \Lambda_1(v_1^*, w_1^*)} J_1(u).$$

Then as in Proposition 6.74,

$$d_1(v_1^*, w_1^*) > c_1(v_1^*, w_1^*). \tag{11.38}$$

The argument of (A) showing that $U \le w_1^*$ for $x_1 \le m_1$ likewise proves that $U \ge v_2^*$ for $x_1 \ge m_2 - \ell$. Hence $f_1^*(U) = w_1^*$ for $x_1 \ge m_2 - \ell$, so by (11.3) (i) and (11.32), $\tau_j^1 f_1^*(U) \in \Lambda_1(v_1^*, w_1^*)$. Therefore

$$J_1(f_1^*(U)) = J_1(\tau_j^1 f_1^*(U)) \ge d_1(v_1^*, w_1^*). \tag{11.39}$$

For $v, w \in \mathcal{M}_0$, let

$$Y^*(v, w) = \{u \in \Gamma_1(v, w) \mid u = v \text{ for large negative } x_1,$$
$$\text{and } u = w \text{ for large positive } x_1\}$$

and set

$$c^*(v, w) = \inf_{u \in Y^*(v,w)} J_1(u). \tag{11.40}$$

Then as in (11.8), by (11.39),

$$c_{m,\ell}^* = J_1(U) = \sum_1^3 J_1(f_i^*(U)) \ge d_1(v_1^*, w_1^*) + c^*(w_1^*, v_2^*) + J_1(f_3^*(U)). \tag{11.41}$$

Note that $f_3^*(U) = U$ for $x_1 \ge m_2 - \ell$, $f_3^*(U) = v_2^*$ for $x_1 \le m_1$, and $f_3^*(U)$ is near w_2^* in X_{i_2}. Hence modifying U in X_{i_2} so that the modified function equals w_2^* in T_{i_2} readily yields

$$J_1(f_3^*(U)) \ge c_1(v_2^*, w_2^*) + c_1(w_2^*, v_2^*) - \kappa(\sigma) \tag{11.42}$$

as in part (D) of the proof of Theorem 6.8.

On the other hand, for $m_2 - m_1$ and $m_3 - m_2$ sufficiently large, as in Chapters 7 and 10, we find an upper bound for $c_{m,\ell}^*$ of the form

$$c_{m,\ell}^* \le c_1(v_1^*, w_1^*) + c^*(w_1^*, v_2^*) + c_1(v_2^*, w_2^*) + c_1(w_2^*, v_2^*) + 4\epsilon, \tag{11.43}$$

where $\epsilon \to 0$ as $m_2 - m_1, m_3 - m_2 \to \infty$. Combining (11.41)–(11.43) shows

$$d_1(v_1^*, w_1^*) - c_1(v_1^*, w_1^*) \le 4\epsilon + \kappa(\sigma). \tag{11.44}$$

Choosing $m_2 - m_1$ and $m_3 - m_2$ so large and σ so small that

$$4\epsilon + \kappa(\sigma) < \frac{1}{2} \min_{i=1,2} (d_1(v_i^*, w_i^*) - c_1(v_i^*, w_i^*), d_1(w_2^*, v_2^*) - c_1(w_2^*, v_2^*)) \tag{11.45}$$

shows that (11.44) and (11.45) are not compatible.

Thus case (b) is not possible and U satisfies (11.2) (i) with strict inequality. A similar argument applies to get (11.2) (iii). Thus it remains only to verify (11.2) (ii). As earlier, this reduces to treating either $j < \ell_2 - 2$ or $j > \ell_2 + 2$. But both of these possibilities can be excluded as was case (b) above.

Proof of (D). At this point we know that U is C^2 and is a solution of (PDE) outside of the constraint regions and even in X_{i_j}, $j = 1, 2, 3$. To handle the constraint regions, suppose first that z satisfies $m_1 - \ell \le z_1 < m_1$. Recall that $v_1^* \le u_k$, $U \le w_1^*$ for $x_1 \le m_1$. Take $r > 0$ such that $B_r(z) \subset \{x_1 < m_1\}$ and smooth φ with support in $B_r(z)$. Then for $|t|$ small, $u_k + t\varphi$ satisfies (11.2) but not necessarily (11.3). However, arguing as following (11.17), where now $i_1 + 3$ is replaced by m_1, shows that

$$\chi_k = \min(\max(u_k + t\varphi, v_1^*), w_1^*) \in Y_{m,\ell}^*$$

and

$$c_{m,\ell}^* \le J_1(\chi_k) \le J_1(u_k + t\varphi).$$

Therefore by Proposition 2.64, U satisfies (PDE) in $B_r(z)$. A similar argument holds for $z_1 > m_2 - \ell$.

The proof of (D) has now been reduced to showing that U is in C^2 and satisfies (PDE) in a neighborhood of $x_1 = m_1$ and $x_1 = m_2 - \ell$. These two remaining cases require regularity arguments such as arise in the study of obstacle problems.

To complete the proof, since the two cases are handled similarly, we will treat the case of $x_1 = m_1$. Translating variables, we can assume $m_1 = 0$. Set $\Omega = (-1, 1) \times \mathbb{T}^{n-1}$. We know $U \in W^{1,2}(\Omega)$ and is a solution of (PDE) in Ω for $x_1 \ne 0$. The remainder of the argument will be divided into five steps: (E) U is defined everywhere in Ω and is upper semicontinuous (usc); (F) completion of the proof when $U < w_1^*$ for $x_1 = 0$; (G) $U \in C(\Omega)$; (H) U is Lipschitz continuous in Ω with Lipschitz constant depending only on $\|F_u\|_{L^\infty(\mathbb{T}^{n+1})}$; and finally (I) for $m_2 - m_1$, $m_3 - m_2$ sufficiently large, $U < w_1^*$ on $x_1 = 0$. Steps (E)–(H) are based on material that Misha Feldman provided us and for which we are grateful.

Proof of (E). Set

$$\begin{aligned}\mathcal{A} = \{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}^{n-1}) | v_1^* \le u \le w_2^* \text{ for } x \in \Omega, v_1^* \le u \le w_1^* \text{ in } \overline{T}_{-1},\\ \|u - v_1^*\|_{L^2(T_{-1})} \le \rho_1, \text{and } u = U \text{ for } |x_1| \ge 1\}.\end{aligned}$$

By the definition of $c_{m,\ell}^*$,

$$J_1(U) = \inf_{u \in \mathcal{A}} J_1(u),$$

or equivalently,

$$I(U) \equiv \int_\Omega L(U)dx = \inf_{u \in \mathcal{A}} I(u). \tag{11.46}$$

Let $z \in \Omega$ and $r > 0$ be such that $B_r(z) \subset \Omega$. Suppose φ is smooth with support in $\overline{B_r(z)}$. Then for $t > 0$, $\min(U + t\varphi, v_1^*) \in \Gamma_1(v_1^*)$, so as in earlier arguments,

$$\begin{aligned} J_1(U + t\varphi) &= J_1(\max(U + t\varphi, v_1^*)) + J_1(\min(U + t\varphi, v_1^*)) \\ &\geq J_1(\max(U + t\varphi, v_1^*)). \end{aligned} \tag{11.47}$$

Further requiring that $\varphi \leq 0$ yields $\max(U + t\varphi, v_1^*) \in Y_{m,\ell}^*$, so by (11.47),

$$J_1(U + t\varphi) \geq J_1(U)$$

or

$$I(U + t\varphi) \geq I(U).$$

Consequently, setting $g(x) \equiv F_u(x, U(x))$ leads to

$$I'(U)\varphi = \int_\Omega (\nabla U \cdot \nabla \varphi + g\varphi)dx \geq 0 \tag{11.48}$$

for all such r, z, φ. Here I' denotes the Fréchet derivative of I. By (11.48), U is a weak subsolution of (PDE) in Ω.

Next for $\epsilon > 0$, let $\eta_\epsilon(x)$ be a family of mollifiers, i.e., $\eta_\epsilon \in C^\infty(\mathbb{R}^n)$ with the support of η_ϵ, supp η_ϵ contained in $B_\epsilon(0)$ and

$$\int_{\mathbb{R}^n} \eta_\epsilon(x)dx = 1.$$

Set $U_\epsilon = U * \eta_\epsilon$, the convolution of U and η_ϵ. Similarly set $g_\epsilon = g * \eta_\epsilon$. Since U and g are defined in a neighborhood of Ω, U_ϵ and g_ϵ are defined and in $C^\infty(\overline{\Omega})$. Let $\zeta \in C_0^\infty(\Omega)$ with $\zeta \leq 0$. By (11.48),

$$\begin{aligned} \int_\Omega \nabla U_\epsilon \cdot \nabla \zeta dx &= \int_\Omega \left(\int_{B_\epsilon(0)} \nabla U(x-y)\eta_\epsilon(y)dy \right) \cdot \nabla \zeta(x)dx \\ &\geq -\int_\Omega \left(\int_{B_\epsilon(0)} g(x-y)\eta_\epsilon(y)dy \right) \zeta(x)dx \\ &= -\int_\Omega g_\epsilon \zeta \, dx. \end{aligned} \tag{11.49}$$

Since U_ϵ and g_ϵ are C^∞ functions, (11.49) implies

$$-\Delta U_\epsilon + g_\epsilon \leq 0 \ \text{ in } \ \Omega. \tag{11.50}$$

Fix $z \in \Omega$ and define

$$G(r) \equiv G(r,z,\epsilon) = \frac{1}{|B_r(z)|}\int_{B_r(z)} U_\epsilon(y)dy = \frac{1}{|B_1(0)|}\int_{B_1(0)} U_\epsilon(z+rx)dx.$$

Then with ν denoting the outward-pointing normal to $\partial B_1(0)$,

$$\begin{aligned} |B_1(0)|G'(r) &= \int_{B_1(0)} x \cdot \nabla U_\epsilon(z+rx)dx \\ &= -\frac{1}{2}\int_{B_1(0)} |x|^2 r \Delta U_\epsilon(z+rx)dx + \frac{1}{2}\int_{\partial B_1(0)} \frac{\partial U_\epsilon}{\partial \nu} dH^{n-1} \\ &= \frac{r}{2}\int_{B_1(0)} (1-|x|^2)\Delta U_\epsilon \, dx \\ &\geq \frac{r}{2}\int_{B_1(0)} (1-|x|^2) g_\epsilon \, dx. \end{aligned} \tag{11.51}$$

Now

$$\left|\int_{B_1(0)} (1-|x|^2) g_\epsilon(z+rx)dx\right| \leq \|g_\epsilon\|_{L^\infty(\Omega)}|B_1(0)| \leq \|F_u\|_{L^\infty(\mathbb{T}^n+1)}|B_1(0)|, \tag{11.52}$$

so by (11.51)–(11.52),

$$G'(r) \geq -2rK,$$

or $G(r,z,\epsilon) + Kr^2$ is nondecreasing in r, where K depends on $\|F_u\|_{L^\infty(\mathbb{T}^n+1)}$. Letting $\epsilon \to 0$ and noting that $U_\epsilon \to U$ in L^1_{loc}, we conclude

$$G(r,z,0) \equiv \frac{1}{|B_r(z)|}\int_{B_r(z)} U \, dx,$$

and $G(r,z,0) + Kr^2$ is nondecreasing. This implies that

$$\lim_{r \to 0+} G(r,z,0) \tag{11.53}$$

exists for all $z \in \Omega$. By the Lebesgue differentiation theorem, this limit exists a.e. and equals $U(z)$. Thus (11.53) can be used to define U for all $z \in \Omega$. In particular, it shows that

$$U(z) \leq \frac{1}{|B_r(z)|}\int_{B_r(z)} U \, dx + Kr^2 \tag{11.54}$$

whenever $B_r(z) \subset \Omega$.

Next we obtain some further regularity for U. See also Caffarelli [28], on which the following result and (G) and (H) are based.

Proposition 11.55. *U is usc in Ω.*

Let $z \in \Omega$, $\overline{B_r(z)} \subset \Omega$, and $(z_k) \subset \Omega$ with $z_k \to z$ as $k \to \infty$. Then $\overline{B_r(z_k)} \subset \Omega$ for k large and by (11.54),

$$\begin{aligned}\overline{\lim_{k\to\infty}}\, U(z_k) &\le \overline{\lim_{k\to\infty}} \left(\frac{1}{|B_r(z_k)|} \int_{B_r(z_k)} U\, dx + Kr^2 \right) \\ &\le \overline{\lim_{k\to\infty}} \frac{1}{|B_r(z)|} \left(\int_{B_r(z_k)} U\, dx - \int_{B_r(z)} U\, dx \right) \\ &\quad + \frac{1}{|B_r(z)|} \int_{B_r(z)} U\, dx + Kr^2 \\ &= \frac{1}{|B_r(z)|} \int_{B_r(z)} U\, dx + Kr^2. \end{aligned} \tag{11.56}$$

Now letting $r \to 0$, (11.56) yields

$$\overline{\lim_{k\to\infty}}\, U(z_k) \le U(z)$$

so U is usc in Ω.

Proof of (F). Let $T = \{x \in \Omega \mid U(x) < w_1^*(x)\}$. Since U is usc, T is open. Suppose $U(z) < w_1^*(z)$ for $z \in \Omega$ with $z_1 = 0$. Then $z \in T$ and there is an $r = r(z)$ with $B_r(z) \subset T$. For any smooth $\psi \ge 0$ with support in $B_r(z)$, $U + t\psi \in Y^*_{m,\ell}$ for $t > 0$ small. Hence $I'(U)\psi \ge 0$ for all such ψ. But by (11.48), $I'(U)\psi \le 0$ for all such ψ, so $I'(U)\psi = 0$, and elliptic regularity theory implies that U is a solution of (PDE) in $B_r(z)$.

Proof of (G). Some preliminaries are needed here. It is convenient to replace the minimization problem in $\mathcal{A}$ by a variant of the obstacle problem. Define a function W in $\overline{\Omega}$ via

$$W = \begin{cases} w_1^* & \text{in } \overline{T}_{-1}, \\ \Phi & \text{in } \overline{T}_0, \end{cases}$$

where Φ satisfies

$$\begin{aligned} -\Delta\Phi + g &= 0 \text{ in } \Omega \cap T_0, \\ \Phi|_{x_1=0} = w_1^*;\ \Phi|_{x_1=1} &= U, \end{aligned} \tag{11.57}$$

and g is as in (E). Since $g \in L^\infty$ and $w_1^* \in C^{2,\alpha}$ for any $\alpha \in (0,1)$, the L^p elliptic theory implies that there is a unique $\Phi \in W^{2,p}(\overline{T}_0)$ satisfying (11.57) for any $p > 1$. Since

$$\begin{aligned} -\Delta(U-\Phi) &= 0 \ \text{ in } \ T_0, \\ U-\Phi &= U - w_1^* \le 0 \ \text{ on } \ x_1 = 0, \\ U - \Phi &= 0 \ \text{ on } \ x_1 = 1, \end{aligned} \tag{11.58}$$

by the weak form of the maximum principle (see e.g. Theorem 8.1 of Gilbarg and Trudinger [29]),

$$\max_{T_0}(U-\Phi) \le \max_{\partial T_0}(U-\Phi)^+ = 0,$$

i.e., $U \le \Phi$ in T_0, and therefore

$$U \le W \ \text{ in } \ \Omega. \tag{11.59}$$

Define

$$\begin{aligned} \mathcal{A}^* = \{u \in W^{1,2}_{\text{loc}}(\mathbb{R}\times\mathbb{T}^{n-1}) | v_1^* \le u \le W \ \text{ in } \ \Omega, \\ \|u - v_1^*\|_{L^2(T_{-1})} \le \rho_1, \ \text{ and } \ u = U \ \text{ if } \ |x_1| \ge 1\}. \end{aligned}$$

Then $\mathcal{A}^* \subset \mathcal{A}$, since $\Phi \le w_2^*$ in Ω. Therefore

$$\inf_{\mathcal{A}} J_1 \le \inf_{\mathcal{A}^*} J_1.$$

But by (11.59), $U \in \mathcal{A}^*$, so

$$\inf_{\mathcal{A}^*} J_1 = \inf_{\mathcal{A}} J_1 = J_1(U). \tag{11.60}$$

Some further remarks about the regularity of W are needed. Since $g \in L^\infty(\overline{T}_0) \cap C^{2,\alpha}(T_0)$ and $\Phi = w_1^* \in C^{2,\alpha}$ on $x_1 = 0$ for any $\alpha \in (0,1)$, the L^p regularity theory implies $\Phi \in W^{2,p}([0,\frac{1}{2}] \times \mathbb{T}^{n-1}) \cap C^{2,\alpha}(T_0)$ and in particular $\Phi \in C^1([0,\frac{1}{2}] \times \mathbb{T}^{n-1})$. Moreover, $W = w_1^* \in C^{2,\alpha}(\overline{T}_{-1})$. This readily implies that W is Lipschitz continuous in, e.g., $[-\frac{1}{2},\frac{1}{2}] \times \mathbb{T}^{n-1}$. An estimate is needed for the Lipschitz constant of W in this region. For $x \in T_{-1}$, $W = w_1^* \in C^2(\mathbb{R}\times\mathbb{T}^{n-1})$, so the L^p theory easily implies $\|W\|_{C^1(T_{-1})} \le \overline{M}$, where $\overline{M}$ depends on $\|w_1^*\|_{L^\infty(T_0)}$ and $\|F_u\|_{L^\infty(\mathbb{T}^{n-1})}$. In T_0, $W = \Phi$ with $\Phi = w_1^*$ smooth at $x_1 = 0$. Therefore (11.57) and the L^p estimates give an upper bound for $\|W\|_{C^1([0,\frac{1}{2}]\times\mathbb{T}^{n-1})}$ in terms of $\|w_1^*\|_{C^2(\mathbb{R}\times\mathbb{T}^{n-1})}$ and $\|F_u\|_{L^\infty(\mathbb{T}^{n+1})}$. Thus for $x, y \in [-\frac{1}{2},\frac{1}{2}] \times \mathbb{T}^{n-1}$, we have

$$|W(x) - W(y)| \le \overline{M}|x-y|. \tag{11.61}$$

Next observe that since U is usc and W is continuous, $\{U < W\}$ is open in Ω. Therefore $\{U = W\}$ is relatively closed in Ω. If $x_1 < 0$, $U < w_1^* = W$. Suppose there is some x with $x_1 > 0$ and $U(x) = W(x)$. Since Φ is $C^{2,\alpha}$ near x, (11.58)–(11.59) and the maximum principle imply that $U \equiv W$ in T_0 and therefore on $x_1 = 0$. Thus for this case $\{U = W\} = \overline{T}_0$. Otherwise, $\{U = W\} \subset \{x_1 = 0\}$. In any event, with the aid of the argument of Proposition 2.64, the subset of Ω where (PDE) is not satisfied classically is $S \equiv \{U = W\} \cap \{x_1 = 0\}$. Viewing $\Delta U - g$ as a positive measure μ (via (11.48)), supp μ is contained in S.

Now we are ready for:

Proposition 11.62. $U \in C(\Omega)$.

Proof. By what is already known, the lower semicontinuity of U need only be verified at points $z \in S$. Thus let $(z_k) \subset \Omega$ and $z_k \to z$ as $k \to \infty$. Since $U = W = w_1^*$ on S, we need only check sequences with $z_k \notin S$. Choose a point $y_k \in S$ closest to z_k. Then

$$\delta_k \equiv |z_k - y_k| \le |z_k - z| \to 0$$

and

$$U(y_k) = w_1^*(y_k) \to w_1^*(z) = U(z)$$

as $k \to \infty$. Since U is usc, for any $\epsilon > 0$, there is a $\rho = \rho(\epsilon, z) > 0$ such that $|z - x| \le \rho$ implies $U(x) \le U(z) + \epsilon$. By (11.54),

$$U(y_k) \le \frac{1}{|B_{2\delta_k}(y_k)|} \int_{B_{2\delta_k}(y_k)} U(x)dx + K(2\delta_k)^2. \tag{11.63}$$

It can be assumed that $B_{2\delta_k}(y_k) \subset B_\rho(z)$ for all large k. Therefore by (11.63),

$$U(y_k) \le \frac{1}{|B_{2\delta_k}(y_k)|} \left[\int_{B_{2\delta_k}(y_k) \setminus B_{\delta_k}(z_k)} (U(z) + \epsilon)dx + \int_{B_{\delta_k}(z_k)} U(x)dx \right] + K(2\delta_k)^2. \tag{11.64}$$

Since U satisfies (PDE) in $B_{\delta_k}(z_k)$, by (11.51) with $z = z_k$ and U_ϵ replaced by U,

$$|B_1(0)|G'(r) = -\frac{r}{2} \int_{B_1(0)} (1 - |x|^2) g \, dx.$$

Therefore

$$|G'(r)| \le \frac{r}{2} \|F_u\|_{L^\infty(\mathbb{T}^{n+1})} = 2Kr,$$

so

$$|G(\delta_k, z_k, 0) - U(z_k)| \le \int_0^{\delta_k} |G'(r)|dr \le K\delta_k^2. \tag{11.65}$$

By (11.64)–(11.65) and (11.54) again, for all large k,

$$U(y_k) \le \frac{2^n - 1}{2^n}(U(z) + \epsilon) + \frac{1}{2^n}(U(z_k) + K\delta_k^2) + K(2\delta_k)^2.$$

Hence

$$\underline{\lim}_{k\to\infty} U(y_k) = U(z) \le \frac{2^n - 1}{2^n}(U(z) + \epsilon) + \frac{1}{2^n}\underline{\lim}_{k\to\infty} U(z_k). \tag{11.66}$$

Since ϵ is arbitrary, (11.66) implies

$$\overline{\lim}_{k\to\infty} U(z_k) \le U(z) \le \underline{\lim}_{k\to\infty} U(z_k),$$

i.e., U is lower semicontinuous and therefore continuous at z.

Proof of (H). The goal here is to prove:

Proposition 11.67. *U is Lipschitz continuous in Ω with Lipschitz constant depending only on $\|F_u\|_{L^\infty(\mathbb{T}^{n+1})}$.*

To prove Proposition 11.67, it suffices to get a uniform upper bound, M^*, for ∇U in each of T_{-1} and T_0. Then for any $x, y \in T_0$ and γ a line segment joining them,

$$|U(x) - U(y)| = \left|\int_\gamma \nabla U \cdot dx\right| \le M^*|x - y|.$$

The continuity of U then shows that this estimate holds for all $x, y \in \overline{T}_0$, and the same is true for $x, y \in \overline{T}_{-1}$. Finally, if $x \in \overline{T}_{-1}$ and $y \in \overline{T}_0$, there is a $z \in \{x_1 = 0\} \cap \gamma$ such that

$$\begin{aligned}|U(x) - U(y)| &\le |U(x) - U(z)| + |U(z) - U(y)| \\ &\le M^*(|x - z| + |z - y|) = M^*|x - y|.\end{aligned}$$

The first step in getting the bound for ∇U is:

Proposition 11.68. *Suppose $t \in \mathbb{R}$ and u is a classical solution of*

$$-\Delta u + F_u(x, u + t) = 0, \quad x \in B_r(y).$$

Then

$$\|\nabla u\|_{L^\infty(\bar{B}_{r}2(y))} \leq \frac{M_1}{r}(\|u\|_{L^\infty(B_r(y))} + r^2), \tag{11.69}$$

where M_1 depends only on $\|F_u\|_{L^\infty(\mathbb{T}^{n+1})}$.

Proof. Translating variables for convenience, we can take $y = 0$. For $x \in B_r(0)$, set $\xi = x/r$ and $\overline{u}(\xi) = u(r\xi)$. Then

$$-\Delta\overline{u} + r^2 F_u(r\xi, \overline{u}(\xi) + t) = 0, \quad \xi \in B_1(0),$$

and by the L^p elliptic theory [29], for any $p > 1$,

$$\begin{aligned}\|\overline{u}\|_{W^{2,p}(B_{\frac{1}{2}}(0))} &\leq M_2(\|\overline{u}\|_{L^\infty(B_1(0))} + r^2\|F_u(\cdot, \overline{u} + t)\|_{L^p(B_1(0))})\\ &\leq M_3(\|\overline{u}\|_{L^\infty(B_r(0))} + r^2),\end{aligned} \tag{11.70}$$

where M_3 depends on p and $\|F_u\|_{L^\infty(\mathbb{T}^{n+1})}$. By standard embedding theorems, for $p > n$,

$$\|\nabla\overline{u}\|_{L^\infty(B_{\frac{1}{2}}(0))} \leq M_4\|\overline{u}\|_{W^{2,p}(B_{\frac{1}{2}}(0))}, \tag{11.71}$$

where M_4 depends on p. Fixing $p > n$ and noting that $\nabla\overline{u} = r\nabla u$, (11.70)–(11.71) yield (11.69).

Remark 11.72. For any $\rho \in (0, 1)$, Proposition 11.68 provides an upper bound for $\|\nabla U\|_{L^\infty(\Omega\setminus B_\rho(S))}$ that depends on ρ, $\|F_u\|_{L^\infty(\mathbb{T}^{n+1})}$, and $\|U\|_{L^\infty(\Omega)}$. Since $v_1^* \leq U \leq w_2^*$, the dependence on $\|U\|_{L^\infty(\Omega)}$ can be suppressed. Thus to complete the proof of Proposition 11.67, an upper bound for $\|\nabla U\|_{L^\infty(B_\rho(S))}$ for some suitable ρ is needed. Toward that end, let $z \in S$.

Proposition 11.73. *There are constants $r_0 \leq \frac{1}{2}$ and M_5 depending on $\|F_u\|_{L^\infty(\mathbb{T}^{n+1})}$ (and $\|w_1^*\|_{L^\infty}$) but independent of $z \in S$ such that if $0 < r \leq r_0$,*

$$U(x) \geq W(z) - M_5 r, \quad x \in B_{\frac{r}{4}}(z).$$

Proof. Set $\Psi(x) = U(x) - W(z) - \overline{M}r$, so by (11.59) and (11.61), $\Psi \leq 0$ in $B_r(z)$. Set

$$\Psi = \Psi_1 + \Psi_2 + \Psi_3, \tag{11.74}$$

where $\Psi_1, \Psi_2 \in W^{2,p}(B_r(z))$ for any $p > 1$ are defined to be the solutions of the following PDEs with continuous data.

$$\begin{aligned} -\Delta\Psi_1 + F_u(x, U) &= 0, & x &\in B_r(z),\\ \Psi_1 &= 0, & x &\in \partial B_r(z),\\ -\Delta\Psi_2 &= 0, & x &\in B_r(z),\\ \Psi_2 &= \Psi, & x &\in \partial B_r(z).\end{aligned}$$

Now (11.74) determines Ψ_3.

As in Proposition 11.68, it can be assumed that $z = 0$. Set $h(y) = \Psi_1(ry)$ for $y \in B_1(0)$, so

$$\begin{aligned} -\Delta h + r^2 F_u(ry, U) &= 0, & y &\in B_1(0), \\ h &= 0, & y &\in \partial B_1(0). \end{aligned}$$

Set $\widehat{h} = \widehat{M} r^2(1 - |y|^2)$ with $\widehat{M}$ a constant. Then

$$-\Delta(\widehat{h} \pm h) = 2n\widehat{M} r^2 \mp r^2 F_u(ry, U) \geq 0, \quad y \in B_1(0)$$

if $\widehat{M} \geq \frac{1}{2n} \|F_u\|_{L^\infty(\mathbb{T}^{n+1})}$, and $h = \widehat{h} = 0$ on $\partial B_1(0)$. Hence by a weak version of the maximum principle (see, e.g., Theorem 8.1 of [29], in $B_1(0)$),

$$|h| \leq \widehat{h} \leq \widehat{M} r^2,$$

or

$$\|\Psi_1\|_{L^\infty(B_r(z))} \leq \widehat{M} r^2. \tag{11.75}$$

Since $\Psi_2 \leq 0$ on $\partial B_r(z)$, by the usual maximum principle,

$$\Psi_2 \leq 0, \quad x \in B_r(z). \tag{11.76}$$

Set $\Psi_4 = \Psi_1 + \Psi_2$. Then by (11.75)–(11.76),

$$\Psi_4 \leq \widehat{M} r^2, \quad x \in B_r(z), \tag{11.77}$$

and

$$\begin{aligned} -\Delta\Psi_4 + F_u(x, U) &= 0, & x &\in B_r(z), \\ \Psi_4 &= \Psi, & x &\in \partial B_r(z) \end{aligned}$$

By the line above (11.74) and (11.48),

$$\int_{B_r(z)} (\nabla\Psi \cdot \nabla\varphi + F_u(x, U)\varphi) dx \leq 0 \tag{11.78}$$

for all $\varphi \in W_0^{1,2}(B_r(z))$ with $\varphi \geq 0$, i.e., Ψ is a weak subsolution of

$$-\Delta\Psi + F_u(x, U) = 0$$

in $B_r(z)$. Hence by (11.74), (11.78), and the choice of Ψ_1, Ψ_2,

$$\int_{B_r(z)} \nabla\Psi_3 \cdot \nabla\varphi \leq 0$$

for all φ as above, i.e., Ψ_3 is weakly subharmonic in $B_r(z)$. Since $\Psi_3 = 0$ on $\partial B_r(z)$, by e.g., the weak maximum principle of [29] again,

$$\Psi_3 \leq 0, \quad x \in B_r(z). \tag{11.79}$$

Combining (11.79), (11.76), and (11.75),

$$\Psi \leq \Psi_1 + \Psi_2 = \Psi_4 \leq \Psi_1 \leq \widehat{M} r^2, \quad x \in B_r(z). \tag{11.80}$$

Therefore

$$\Psi_5 = \widehat{M} r^2 - \Psi_4 \geq 0, \quad x \in B_r(z),$$

and

$$-\Delta \Psi_5 - F_u(x, U) = 0, \quad x \in B_r(z).$$

Applying Theorems 8.17–8.18 of [29] to Ψ_5 gives the weak Harnack inequality:

$$\sup_{B_{\frac{r}{4}}(z)} \Psi_5 \leq M_6 \left(\inf_{B_{\frac{r}{4}}(z)} \Psi_5 + \|F_u\|_{L^\infty(\mathbb{T}^n+1)} r^2 \right) \leq M_7 \left(\inf_{B_{\frac{r}{4}}(z)} \Psi_5 + r^2 \right), \tag{11.81}$$

where M_7 depends on $\|F_u\|_{L^\infty(\mathbb{T}^n+1)}$. Since $z \in S$, by (11.79) and (11.74),

$$\Psi_4(z) = \Psi(z) - \Psi_3(z) \geq U(z) - W(z) - \overline{M} r = -\overline{M} r. \tag{11.82}$$

Hence for $x \in B_{\frac{r}{4}}(z)$, by (11.81)–(11.82),

$$\Psi_5(x) = \widehat{M} r^2 - \Psi_4(x) \leq M_7(\widehat{M} r^2 - \Psi_4(z) + r^2) \leq M_7(\widehat{M} r^2 + \overline{M} r + r^2).$$

Consequently, for $r_0 = r_0(\|F_u\|_{L^\infty(\mathbb{T}^n+1)})$ sufficiently small,

$$\Psi_4(x) \geq -M_8 r, \tag{11.83}$$

where M_8 depends on $\|F_u\|_{L^\infty(\mathbb{T}^n+1)}$.

Next, a similar lower bound will be obtained for Ψ_3. We have already shown that $\Psi_3 = 0$ on $\partial B_r(z)$, $\Psi_3 \leq 0$ in $B_r(z)$, and $-\Delta \Psi_3 = -\Delta U + F_u(x, U) \leq 0$ in $B_r(z)$. Therefore supp $\Delta \Psi_3 \subset$ supp $(\Delta U - F_u(x, U)) \subset S$. Since Ψ_3 is continuous, it has a minimizer in $B_r(z)$, and by the maximum principle, it occurs at some $z^* \in S$. By (11.80),

$$\Psi_3 = \Psi - \Psi_4 \geq \Psi - \widehat{M} r^2.$$

Therefore for all $x \in B_r(z)$,

$$\begin{aligned}\Psi_3(x) \geq \Psi_3(z^*) \geq \Psi(z^*) - \widehat{M} r^2 &= U(z^*) - w_1^*(z) - \overline{M} r - \widehat{M} r^2 \\ &= w_1^*(z^*) - w_1^*(z) - \overline{M} r - \widehat{M} r^2 \\ &\geq -2\overline{M} r - \widehat{M} r^2 \end{aligned} \tag{11.84}$$

via (11.61). Again for r_0 sufficiently small this leads to

$$\Psi_3(x) \geq -M_9 r, \tag{11.85}$$

where M_9 depends on $\|F_u\|_{L^\infty(\mathbb{T}^{n+1})}$. Combining (11.83) and (11.85) gives

$$\Psi(x) = U(x) - W(z) - \overline{M} r \geq -(M_8 + M_9) r,$$

or

$$U(x) \geq W(z) - M_5 r, \quad x \in B_{\frac{r}{4}}(z),$$

where $M_5 = -\overline{M} + M_8 + M_9$, and Proposition 11.73 is proved.

Proof of Proposition 11.67. Note that by (11.59) and (11.61),

$$W(z) - U(x) \geq W(z) - W(x) \geq -\overline{M}|z - x|,$$

so with the aid of Proposition 11.73,

$$|U(x) - W(z)| \leq M_{10} r \tag{11.86}$$

for $z \in S$, $r \leq r_0$, and $x \in B_{\frac{r}{4}}(z)$, where M_{10} depends on $\|F_u\|_{L^\infty(\mathbb{T}^{n+1})}$. Choose any $y \in B_{\frac{r_0}{8}}$ (S). Then there is a $z \in S$ such that $|y - z| = |y - S| \equiv s < \frac{r_0}{8}$. Set $r = 8s$. Then $x \in B_s(y)$ implies $x \in B_{2s}(z) = B_{\frac{r}{4}}(z)$, so (11.86) holds for $x \in B_s(y)$. Restricting x to $B_{\frac{s}{2}}(y)$ and taking $u = U - W(z)$ in (11.69) give

$$\begin{aligned}\|\nabla U\|_{L^\infty(B_{\frac{s}{2}}(y))} &\leq \frac{M_1}{s}(M_{10} r + s^2) \\ &\leq M_1\Big(8M_{10} + \frac{r_0}{8}\Big) \leq M_1\Big(8M_{10} + \frac{1}{16}\Big) \equiv M_{11}, \end{aligned} \tag{11.87}$$

where M_{10} depends on $\|F_u\|_{L^\infty(\mathbb{T}^{n+1})}$. Combining (11.87) with Remark 11.72 with $\rho = \frac{r_0}{8}$ then completes the proof.

Proof of (I). The argument is related to that of (C). We seek to show $U(x) < w_1^*(x)$ for $x_1 = m_1$ and $U(x) > v_2^*(x)$ for $x_1 = m_2 - \ell$. To treat the m_1 case, arguing indirectly, suppose $U(z) = w_1^*(z)$ for some $z = (z_1, \ldots, z_n)$ and $z_1 = m_1$. By the choice of ρ_1, there is a largest $\underline{u}_{m_1} \in \mathcal{M}_1(v_1^*, w_1^*)$ such that $\|\underline{u}_{m_1} - v_1^*\|_{L^2(T_{m_1-1})} < \rho_1$ and a smallest $\overline{u}_{m_1} \in \mathcal{M}_1(v_1^*, w_1^*)$ such that $\|\overline{u}_{m_1} - v_1^*\|_{L^2(T_{m_1-1})} > \rho_1$. Indeed, $\underline{u}_{m_1}, \overline{u}_{m_1}$ are a gap pair in $\mathcal{M}_1(v_1^*, w_1^*)$. Note that $\underline{u}_{m_1} = \tau_{m_1}^1 \underline{u}$ and $\overline{u}_{m_1} = \tau_{m_1}^1 \overline{u}$ with $\underline{u}, \overline{u} \in \mathcal{M}_1(v_1^*, w_1^*)$. Translating variables, we can take $m_1 = 0$.

By Proposition 11.67, there is an $M^* = M^*(\|F_u\|_{L^\infty(\mathbb{T}^{n+1})})$ independent of the minimizer $U \in Y_{m,\ell}^*$ of (11.5) and m such that for $y \in T_{-1}$,

$$U(y) - U(z) \geq -M^*|y - z|.$$

Thus for $|y - z| \leq r$,

$$U(y) \geq w_1^*(z) - M^* r. \tag{11.88}$$

Since $\overline{u} \in \mathcal{M}_1(v_1^*, w_1^*), \bar{u} \in C^2$ and

$$\overline{u}(y) \leq \overline{u}(z) + M_{12}|y - z|, \tag{11.89}$$

where M_{12} depends on $\|F_u\|_{L^\infty(\mathbb{T}^{n+1})}$. Combining (11.88)–(11.89), for $|y - z| \leq r$ and $y \in T_{-1}$,

$$U(y) \geq w_1^*(z) - M^* r > \overline{u}(y),$$

provided that

$$r < \frac{w_1^*(z) - \overline{u}(z)}{M^* + M_{12}}.$$

Thus choosing r such that

$$r < \min_{x \in T_{-1}} \frac{w_1^*(x) - \overline{u}(x)}{M^* + M_{12}}, \tag{11.90}$$

it follows that

$$\overline{u}(y) < U(y) \tag{11.91}$$

for all $y \in B_r(z) \cap T_{-1}$ and all $z \in S$.

Set

$$\Lambda_1^* \equiv \Lambda_1^*(v_1^*, w_1^*) = \{u \in \Gamma_1(v_1^*, w_1^*) \mid (i)\ \|u - v_1^*\|_{L^2(T_{-1})} \leq \rho_1,$$
$$(ii)\ \|u - w_1^*\|_{L^2(B_r(z) \cap T_{-1})} \leq \|\overline{u} - w_1^*\|_{L^2(B_r(z) \cap T_{-1})}\}$$

and

$$d_1^*(v_1^*, w_1^*) = \inf_{u \in \Lambda_1^*} J_1(u). \tag{11.92}$$

Lemma 11.93. $d_1^*(v_1^*, w_1^*) > c_1(v_1^*, w_1^*)$.

Proof. Following the proof of Proposition 6.74 shows that there is a minimizing sequence (u_k) for (11.92) that converges in $W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ to $P \in \widehat{\Gamma}_1(v_1^*, w_1^*)$. If $P \notin \Lambda_1^*$, $J_1(u_k) \geq c_1(v_1^*, w_1^*) + \beta_1$, where $\beta_1 > 0$ depends on ρ_1 and $\|\overline{u} - w_1^*\|_{L^2(B_r(z)\cap T_{-1})}$. On the other hand, if $P \in \Lambda_1^* \subset \Gamma_1(v_1^*, w_1^*)$,

$$J_1(P) = d_1^*(v_1^*, w_1^*) \geq c_1(v_1^*, w_1^*). \tag{11.94}$$

If there is equality in (11.94), $P \in \mathcal{M}_1(v_1^*, w_1^*)$. But then either $P \leq \underline{u}$, in which case constraint (ii) in Λ_1^* fails, or $P \geq \overline{u}$, in which case constraint (i) cannot hold. Therefore $d_1^* > c_1$.

Now to complete the proof of (I), we will show that if $m_2 - m_1$, $m_3 - m_2$ are sufficiently large, $U(z) = w_1^*(z)$ cannot occur for any z with $z_1 = m_1$. The reasoning is essentially the same as that of part (C,) so we will be sketchy. Again it is convenient to translate variables so that $m_1 = 0$.

If $U(z) = w_1^*(z)$, since $f_1^*(U) = U$ for $x_1 \leq 0$, by the choice of r, $f_1^*(U) \in \Lambda_1^*$. Therefore $J_1(f_1^*(U)) \geq d_1^*$ and as in (11.41)–(11.43),

$$\begin{aligned} &d_1^*(v_1^*, w_1^*) + c^*(w_1^*, v_2^*) + c_1(v_2^*, w_2^*) + c_1(w_2^*, v_2^*) - \kappa(\sigma) \\ &\quad \leq c^*_{m,\ell} = J_1(U) \leq c_1(v_1^*, w_1^*) + c^*(w_1^*, v_2^*) \\ &\qquad + c_1(v_2^*, w_2^*) + c_1(w_2^*, v_2^*) + 4\epsilon. \end{aligned} \tag{11.95}$$

Thus choosing m_2, and $m_3 - m_2$ so large and σ so small that

$$4\epsilon + \kappa(\sigma) < \frac{1}{2} \min_{i=1,2}(d_1^*(v_i^*, w_i^*) - c_1(v_i^*, w_i^*)) \tag{11.96}$$

shows that $U(z) = w_1^*(z)$ for $z_1 = 0$ is not possible. Similarly, $U(z) = v_2^*(z)$ for $z_1 = m_2 - \ell$ cannot occur, and Theorem 11.6 is proved.

Remark 11.97. It is possible that there are several gap pairs in $\mathcal{M}_0$ between w_1^* and v_2^*, e.g.,

$$w_1^* \leq \overline{v}_1 < \overline{w}_1 \leq \cdots \leq \overline{v}_j < \overline{w}_j \leq v_2^*.$$

A more refined version of Theorem 11.6 would then give a solution of (PDE) that shadows members of $\mathcal{M}_1(v_1^*, w_1^*)$, $\mathcal{M}(\overline{v}_i, \overline{w}_i)$, $1 \leq i \leq j$, and $\mathcal{M}(w_2^*, v_2^*)$.

Next we will discuss how to extend Theorem 11.6 to the case in which there are k gap pairs where transitions occur. Thus suppose that $v_i^* < w_i^*$, $1 \leq i \leq k$, are gap pairs in $\mathcal{M}_0$. Repetition of pairs is permitted. Suppose that

$$v_1^* \leq v_2^* \geq v_3^* \leq v_4^* \geq \cdots . \tag{11.98}$$

Alternatively, (11.98) could be replaced by

$$v_1^* \geq v_2^* \leq v_3^* \geq v_4^* \leq \cdots .$$

We seek a solution of (PDE) that is heteroclinic from v_1^* to w_k^* if k is even or to v_k^* if k is odd. Moreover, the solution will be required to be close to $w_2^*, v_3^*, w_4^*, \ldots$ in intermediate regions. Choose $m \in \mathbb{Z}^k$ with $m_{i+1} > m_i$, $1 \leq i \leq k-1$, and $l \in \mathbb{N}$. Set

$$v_* = \min_{1\leq i\leq k} v_i^* \textit{ and } w^* = \max_{1\leq i\leq k} w_i^*$$

In the spirit of (11.1)–(11.3), as the class of admissible functions we take

$$\begin{aligned} Y_{m,\ell}^* = \{u \in W_{\text{loc}}^{1,2}(\mathbb{R}\times\mathbb{T}^{n-1}) \mid v_* \leq u \leq w^* \\ \text{and } u \text{ satisfies (11.99)–(11.101)}\}, \end{aligned}$$

where

$$\begin{cases} \text{(i)} \ \|u - v_1^*\|_{L^2(T_i)} \to 0, & i \to -\infty, \\ \text{(ii)} \ \|u - \phi\|_{L^2(T_i)} \to 0, & i \to \infty, \end{cases} \tag{11.99}$$

and $\phi = w_k^*$ if k is even and $\phi = v_k^*$ if k is odd;

$$\begin{cases} \text{(i)} \ \|u - v_1^*\|_{L^2(T_i)} \leq \rho_1, \ m_1 - \ell \leq i \leq m_1 - 1, \\ \text{(ii)} \ \|u - \phi\|_{L^2(T_i)} \leq \rho_k, \ \ m_k \leq i \leq m_k + \ell - 1, \\ \text{(iii)} \ \|u - \phi_j\|_{L^2(T_i)} \leq \rho_j, \, m_j - \ell \leq i \leq m_j + \ell - 1, \end{cases} \tag{11.100}$$

where in (ii), ϕ is as in (11.99) and in (iii), for $2 \leq j \leq k-1$, $\phi_j = w_j^*$ if j is even, $\phi_j = v_j^*$ if j is odd.

Lastly,

$$\begin{cases} \text{(i)} \ v_1^* \leq u \leq w_1^*, \ m_1 - \ell \leq x_1 \leq m_1, \\ \text{(ii)} \ u \geq v_k^*, \qquad\quad m_k \leq x_1 \leq m_k + \ell, \\ \text{(iii)} \ v_j^* \leq u \leq w_j^*, \ m_j - \ell \leq x_1 \leq m_j + \ell, \end{cases} \tag{11.101}$$

where $2 \leq j \leq k-1$ in (iii). The constants ρ_i are as in (11.4), and for $1 \leq j \leq k$ satisfy

$$\rho_j \in \left(0, \frac{1}{2}\|w_j^* - v_j^*\|_{L^2(T_0)}\right) \backslash \{\|u - \psi\|_{L^2(T_0)} | u \in \mathcal{M}_1(v_j^*, w_j^*) \cup \mathcal{M}_1(w_j^*, v_j^*)\}, \tag{11.102}$$

where $\psi = w_j^*$ if j is even and $\psi = v_j^*$ if j is odd.

Define

$$c^*_{m,l} = \inf_{u \in Y^*_{m,l}} J_1(u). \quad (11.103)$$

Then analogously to Theorem 11.6, we have:

Theorem 11.104. *Suppose* $(F_1) - (F_2)$ *hold,* (v^*_i, w^*_i) *are gap pairs satisfying (11.98),* $1 \le i \le k$*, and* $(*)_1$ *holds for* $\cup_{i=1}^{i=k}(\mathcal{M}_1(v^*_i, w^*_i) \cup \mathcal{M}_1(w^*_i, v^*_i))$*. Then for* l *sufficiently large, there is a* $U \in Y^*_{m,l}$ *such that* $J_1(U) = c^*_{m,l}$*. Moreover, for* $m_{i+1} - m_i$ *sufficiently large,* $1 \le i \le k$*, any such* U *is a classical solution of (PDE).*

Proof. Since the proof is close to that of Theorem 11.6, we will be sketchy here. As earlier, a minimizing sequence for (11.103) can be assumed to converge to $U \in \Gamma_1(v_*, w^*)$ satisfying (11.100)–(11.101). Set

$$g_j(U) = \max(\min(U, w^*_j), v^*_j).$$

Then by (11.101), $g_j(U) = U$ in the j-th constraint region, and following (11.8)–(11.22), U satisfies (PDE) in some region X_{i_j}, $1 \le j \le k$, provided that

$$l \ge l_0(\sigma, M + kK_1) \quad (11.105)$$

and

$$0 < \sigma < \min_{1 \le j \le k}\left(\frac{\sqrt{5}}{2}\left(\|w^*_j - v^*_j\|_{L^2(T_0)}, \rho_j\right)\right). \quad (11.106)$$

Continuing to follow the proof of Theorem 11.6, the proof of Theorem 11.104 reduces to showing that U satisfies (PDE) in a neighborhood of $x_1 = m_1$, $m_j - l$, $m_j + l$, and $m_k - l$ for $1 \le j \le k$. This in turn is a consequence of slight modifications of (E)–(I) of the earlier proof.

Remark 11.107. Theorem 11.104 does not suffice to obtain solutions that undergo an infinite number of transitions. Indeed, as (11.105) shows, l and therefore σ will depend on k, the number of prescribed transitions. Thus this remains an open question. We suspect that this can be approached as in Chapter 10.

Part III
Solutions of (PDE) Defined on $\mathbb{R}^2 \times \mathbb{T}^{n-2}$

In Chapters 6–11, solutions of (PDE) on $\mathbb{R} \times \mathbb{T}^{n-1}$ have been obtained using the basic solutions of Chapters 1–3 as building blocks. Our final two Chapters 12–13, establish the existence of solutions of (PDE) defined on $\mathbb{R}^2 \times \mathbb{T}^{n-2}$. The basic solutions will be those of Chapter 4.

Chapter 12
A Class of Strictly 1-Monotone Infinite Transition Solutions of (PDE)

Following the ideas and methods of Chapters 6–11, solutions of (PDE) on $\mathbb{R}^2 \times \mathbb{T}^{n-2}$ that undergo a finite number of transitions can readily be constructed. For example in the spirit of Chapters 9–10 solutions that are strictly 1-monotone and heteroclinic in x_2 from v_1 to w_k with v_1, w_1 and v_k, w_k gap pairs can be obtained. Likewise, as in Chapters 6–8, nonmonotone heteroclinics in x_2 from v_1 to w_1 (or homoclinic to v_1) with multiple transitions can be found. Our main goal here, however, is to find a new class of solutions that undergo an infinite number of transitions and are heteroclinic between the members of a gap pair in $\mathcal{M}_0$, both in x_1 and in x_2.

To describe these solutions more fully, suppose $(*)_0$ holds and v_0 and w_0 are a gap pair in $\mathcal{M}_0$. Then by Theorem 3.2, $\mathcal{M}_1(v_0, w_0) \neq \emptyset$. For simplicity we assume that

$$\mathcal{M}_1(v_0, w_0) = \{\tau^1_{-j} v_1 \equiv \psi_j \mid j \in \mathbb{Z}\} \tag{$\mathcal{M}_1$}$$

for any $v_1 \in \mathcal{M}_1(v_0, w_0)$. In particular, we choose v_1 such that

$$0 < \psi_0 - v_0 \leq \frac{w_0 - v_0}{3} \ \text{ in } \ T_0. \tag{12.1}$$

By $(\mathcal{M}_1)$ and Theorem 4.40, $\mathcal{M}_2(\psi_j, \psi_{j+1}) \neq \emptyset$ for all $j \in \mathbb{Z}$. Again for convenience assume that

$$\mathcal{M}_2(\psi_0, \psi_1) = \{\tau^2_{-j} v_2 \equiv h_j | j \in \mathbb{Z}\} \tag{$\mathcal{M}_2$}$$

for any $v_2 \in \mathcal{M}_2(\psi_0, \psi_1)$. By $(\mathcal{M}_2)$, for any $k \in \mathbb{Z}$,

$$\mathcal{M}_2(\psi_k, \psi_{k+1}) = \tau^1_{-k} \mathcal{M}_2(\psi_0, \psi_1) = \{\tau^1_{-k} h_j | j \in \mathbb{Z}\}. \tag{12.2}$$

Let $i \in \mathbb{Z}$ and $k \in \mathbb{N}$. Following the spirit of Chapters 9–10, we will first show that there is a solution $U = U_{i,k}$ of (PDE) that is heteroclinic from ψ_i to

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ψ_{i+k}, is 1-monotone in x_2 (and x_1), and shadows members of $\mathcal{M}_2(\psi_j, \psi_{j+1})$, $i \le j \le i+k-1$. Then choosing, e.g., $i = -q$, $k = 2q-1$, working in an appropriate subclass of the above $U_{i,k}$'s and letting $q \to \infty$, we will find a solution of (PDE) that is heteroclinic from v_0 to w_0 in x_2 and x_1 and shadows members of $\mathcal{M}_2(\psi_j, \psi_{j+1})$ for all $j \in \mathbb{Z}$. For the proofs, we will modify closely related results in the setting of an Allen–Cahn model equation [30, 31].

To begin, $U_{i,k}$ will be obtained by minimizing J_2 over a suitable class of admissible functions $Z(i,k)$. To define $Z(i,k)$, let $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2 \times \mathbb{T}^{n-2})$. Using the notation of Chapter 10, with the caveat that now we are dealing with $\mathbb{R}^2 \times \mathbb{T}^{n-2}$ rather than $\mathbb{R} \times \mathbb{T}^{n-1}$, for $j \in \mathbb{Z}$, set

$$f_j(u) = \min(\max(u, \psi_j), \psi_{j+1}),$$

so $\psi_j \le f_j(u) \le \psi_{j+1}$ and $f_j(u) \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2 \times \mathbb{T}^{n-2})$. Choose $s_0, t_0 \in \mathbb{R}$ such that

$$\int_{T_0} \psi_0 dx < s_0 < t_0 < \int_{T_0} \psi_1 \, dx, \tag{12.3}$$

$$\left\{ h \in \mathcal{M}_2(\psi_0, \psi_1) | \; s_0 < \int_{T_0} h \, dx < t_0 \right\} = \{h_0\}, \tag{12.4}$$

and

$$s_0 \ne \int_{T_0} h \, dx \ne t_0 \tag{12.5}$$

for all $h \in \mathcal{M}_2(\psi_0, \psi_1)$. Since $\psi_0 < h < \psi_1$ for each $h \in \mathcal{M}_2(\psi_0, \psi_1)$, $(\mathcal{M}_2)$ implies that we can find numbers $s_0 < t_0$ for which (12.3)–(12.5) hold.

Let $m \in \mathbb{Z}^\infty$, i.e., $m = (m_i)_{i \in \mathbb{Z}}$ with $m_{i+1} > m_i$. For any finite set of j's, there are functions u satisfying

$$s_0 \le \int_{T_{-j,m_j}} f_j(u) dx \le t_0, \tag{12.6}$$

where $T_{p,q} = \tau^1_p \tau^2_q T_0$. For $i \in \mathbb{Z}$ and $k \in \mathbb{N}$, define

$$Z(i,k) = \{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2 \times \mathbb{T}^{n-2}) \mid \psi_i \le u \le \psi_{i+k}, u \le \tau^p_{-1} u \;\text{ for }\; p = 1, 2,$$
$$\text{and } \; u \; \text{ satisfies (12.6) for } \; i \le j \le i+k-1\}$$

and set

$$b(i,k) = \inf_{u \in Z(i,k)} J_2(u). \tag{12.7}$$

Then we have:

Theorem 12.8. *Let F satisfy (F_1)–(F_2), $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$ hold, and $N, R \in \mathbb{N}$. Then m can be chosen so that for each $i \in \mathbb{Z}$ and $k \in \mathbb{N}$, there is a function $U = U_{i,k} \in Z(i,k)$ such that $J_2(U) = b(i,k)$. Any such U is a classical solution of (PDE) satisfying*

$$U < \tau^p_{-1} U, \quad p = 1, 2, \tag{12.9}$$

$$\|U - \psi_i\|_{W^{1,2}(S_\ell)} \to 0, \quad \ell \to -\infty,$$
$$\|U - \psi_{i+k}\|_{W^{1,2}(S_\ell)} \to 0, \quad \ell \to \infty, \tag{12.10}$$

and for $i \le j \le i + k - 1$,

$$f_j(U) = U \quad on \quad \tau_j^1 \tau_{-m_j}^2 A_{N,R}, \tag{12.11}$$

where $A_{N,R} = [-N, N] \times [-R, R] \times \mathbb{T}^{n-2}$.

Remark 12.12. While there is a formal similarity of Theorem 12.8 with Theorem 10.27, the current situation is different and seems more delicate, since now we are dealing with a single gap pair $v_0, w_0 \in \mathcal{M}_0$ rather than multiple such pairs as in Theorem 10.27. For example, for Theorems 10.27 and 10.63 it is possible to allow the numbers m_i to be a uniform (and large) distance apart: $m_{i+1} = m_i + \nu$. However, we are unable to do that here. Instead, the freedom in choosing the m_i's will be employed to require $m_{i+1} - m_i \to \infty$ as $|i| \to \infty$.

In order to prove Theorem 12.8, some preliminaries are needed. Let

$$\Lambda_2(\psi_0, \psi_1) = \left\{ u \in \Gamma_2(\psi_0, \psi_1) | u \le \tau_{-1}^2 u \;\; and \;\; \int_{T_{0,0}} u \, dx \in \{s_0, t_0\} \right\}.$$

Set

$$d_2(\psi_0, \psi_1) = \inf_{u \in \Lambda_2(\psi_0, \psi_1)} J_2(u). \tag{12.13}$$

Then we have:

Proposition 12.14. $d_2(\psi_0, \psi_1) > c_2(\psi_0, \psi_1)$.

Proof. The proof follows that of Proposition 6.74 and will be omitted.

Remark 12.15. $d_2(\psi_j, \psi_{j+1}) = d_2(\psi_0, \psi_1)$ and $c_2(\psi_j, \psi_{j+1}) = c_2(\psi_0, \psi_1)$ for all $j \in \mathbb{Z}$.

Proposition 12.16. *Let* $r > 0$*. Then there are an* $\ell_1(r) \in \mathbb{N}$ *and* $\Phi \in \Gamma_2(\psi_0, \psi_1)$ *such that*

$$\Phi \le \tau_{-1}^p \Phi, \quad p = 1, 2, \tag{12.17}$$

$$s_0 < \int_{T_{0,0}} \Phi \, dx < t_0, \tag{12.18}$$

$$\Phi = \begin{cases} \psi_0, & x_2 \le -\ell_1, \\ \psi_1, & x_2 \ge \ell_1, \end{cases} \tag{12.19}$$

and

$$J_2(\Phi) \le c_2(\psi_0, \psi_1) + r. \tag{12.20}$$

Proof. The function h_0 satisfies (12.17)–(12.18) and (12.20), and $\|h_0 - \psi_0\|_{W^{1,2}(S_i)} \to 0$ as $i \to -\infty$, $\|h_0 - \psi_1\|_{W^{1,2}(S_i)} \to 0$ as $i \to \infty$. Therefore for large ℓ_1,

$$\Phi = \begin{cases} \psi_0, & x_2 \le -\ell_1, \\ (-\ell_1 + 1 - x_2)\psi_0 + (x_2 + \ell_1)h_0, & -\ell_1 \le x_2 \le -\ell_1 + 1, \\ h_0, & -\ell_1 + 1 \le x_2 \le \ell_1 - 1, \\ (\ell_1 - x_2)h_0 + (x_2 - \ell_1 + 1)\psi_1, & \ell_1 - 1 \le x_2 < \ell_1, \\ \psi_1, & \ell_1 \le x_2, \end{cases}$$

satisfies (12.17)–(12.20).

Corollary 12.21. *For $q \in \mathbb{Z}$, $\Phi_q = \tau^1_{-q}\Phi \in \Gamma_2(\psi_q, \psi_{q+1})$ and satisfies (12.17) and (12.20) as well as (12.18) with $T_{0,0}$ replaced by $T_{0,-q}$ and (12.19) with ψ_0, ψ_1 replaced by ψ_q, ψ_{q+1}.*

One final preliminary is required. It is a version of Proposition 9.20 for the current setting.

Proposition 12.22. *For any $\sigma > 0$, there is a $\delta_2 = \delta_2(\sigma)$ such that whenever $u \in \Gamma_2(\psi_0, \psi_1)$ satisfies $J_2(u) \le c_2(\psi_0, \psi_1) + \delta_2$, there is a $\Psi \in \mathcal{M}_2(\psi_0, \psi_1)$ with*

$$\|u - \Psi\|_{W^{1,2}(\cup_{j=-2}^{2} S_{q+j})} \le \sigma \quad \textit{for all} \quad q \in \mathbb{Z}.$$

Proof. Following the proof of Proposition 9.20, making the natural changes such as replacing $\tau^1_{-\ell_k}$ by $\tau^2_{-\ell_k}$ and (9.22) by a new normalization, such, for instance, as in (4.41), yields the result.

Now we are ready for the

Proof of Theorem 12.8. Set

$$\delta = d_2(\psi_0, \psi_1) - c_2(\psi_0, \psi_1). \tag{12.23}$$

For m, we initially require

$$\begin{aligned} &m_0 = 0,\ m_i = -m_{-i}, \quad \text{and for } i \ge 0, \\ &m_{i+1} - m_i \ge 2\ell_1\left(\frac{\delta}{2^{i+1}}\right) + 2\ell_1\left(\frac{\delta}{2^{i+3}}\right), \end{aligned} \tag{12.24}$$

where ℓ_1 is given by Proposition 12.16. A further restriction on m will be imposed later. Set $\Psi_j = \tau_{m_j}^2 \Phi_j$ with Φ_j as in Corollary 12.21. Then Ψ_j satisfies

$$s_0 < \int_{T_{-j,m_j}} \Psi_j \, dx < t_0, \tag{12.25}$$

$$\Psi_j = \begin{cases} \psi_j, & x_2 \le m_j - \ell_1\left(\dfrac{\delta}{2^{|j|+2}}\right), \\ \\ \psi_{j+1}, & x_2 \ge m_j + \ell_1\left(\dfrac{\delta}{2^{|j|+2}}\right), \end{cases} \tag{12.26}$$

and

$$J(\Psi_j) \le c_2(\psi_0, \psi_1) + \delta/2^{|j|+2}. \tag{12.27}$$

Gluing the functions Ψ_j, $i \le j \le i+k-1$, yields $\Psi \in Z(i, i+k)$ with

$$b(i,k) \le J_2(\Psi) \le kc_2(\psi_0, \psi_1) + 3\delta/4. \tag{12.28}$$

By (12.28) and the argument of Theorem 4.40, if (u_q) is a minimizing sequence for (12.7), there is a $U \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2 \times \mathbb{T}^{n-2})$ such that along a subsequence, $u_q \to U$ in $W^{1,2}(S_\ell)$ for all $\ell \in \mathbb{Z}$, $J_2(U) < \infty$, and $U \in Z(i,k)$. Hence U satisfies (12.10) and $J_2(U) = b(i,k)$ via an earlier argument. Once we have proved that U satisfies (PDE), (12.9) follows as in earlier results, e.g., (c) of Theorem 3.2. Thus it remains to prove that U is a solution of (PDE) and (12.11) holds.

To verify that U satisfies (PDE) requires slight modifications of the proof of Theorem 9.6, so we will be sketchy. Take $z \in \mathbb{R}^2 \times \mathbb{T}^{n-2}$, $p = (p_1, p_2) \in \mathbb{Z}^2$, and set $z_p = z + p_1 e_1 + p_2 e_2$. With these slight changes, follow the earlier proof giving the natural definitions of E_p, I_p, $G(U)$, etc. If $G(U) \in Z(i,k)$, then as in (9.46)–(9.47), U is a solution of (PDE).

To show that $G(U) \in Z(i,k)$, note first that $\psi_i \le G(U) \le \psi_{i+k}$ via the minimality properties of ψ_i and ψ_{i+k} as in (9.43)–(9.44). Next, to verify that $G(U)$ satisfies the constraints (12.6), first observe that U satisfies (12.6) with strict inequality. Indeed, as in (9.39)–(9.41), if for $\sigma \in \{s_0, t_0\}$,

$$\sigma = \int_{T_{-j,m_j}} f_j(U) dx, \tag{12.29}$$

then $f_j(U) \in \Lambda_2(\psi_j, \psi_{j+1})$, so by Proposition 12.14 and Remark 12.15,

$$J_2(f_j(U)) \ge d_2(\psi_0, \psi_1). \tag{12.30}$$

Thus as in (9.41),

$$\begin{aligned} b(i,k) = J_2(U) &= J_2(\min(v_j, U)) + J_2(f_j(U)) + J_2(\max(v_{j+1}, U)) \\ &\ge (k-1)c_2(\psi_0, \psi_1) + d_2(\psi_0, \psi_1). \end{aligned} \tag{12.31}$$

But by (12.31) and (12.28),

$$\frac{3\delta}{4} \geq d_2(\psi_0, \psi_1) - c_1(\psi_0, \psi_1) = \delta, \tag{12.32}$$

a contradiction. Thus U satisfies (12.6) with strict inequality. For $i \leq j \leq i+k-1$, consider

$$D_j \equiv \int_{T_{-j,m_j}} (f_j(G(U)) - f_j(U))dx.$$

The integrand vanishes except possibly for $x \in \cup_{p\in\mathbb{Z}^2} B_r(z_p)$, so the contribution to the integral comes from a set of measure $\leq |B_r(0)|$. Therefore as in (9.49),

$$|D_j| \leq \|\psi_{j+1} - \psi_j\|_{L^\infty(\mathbb{R}^2\times\mathbb{T}^{n-2})}|B_r(0)| \leq |B_r(0)| \to 0 \tag{12.33}$$

as $r \to 0$. Thus choosing $r = r(U)$ small enough, the strict inequality in (12.6) for U implies $G(U)$ satisfies (12.6). Next observe that the argument of (9.50)–(9.54) shows that $G(U) \in Z(i, k)$.

To complete the proof of Theorem 12.8, we will show that (12.11) holds. To do so, we impose a further requirement on m:

$$m_{i+1} - m_i \geq 2\max(\ell_1(\delta/2^{i+1}) + \ell_1(\delta/2^{i+3}), \ell_1(\beta/2^{i+1}) + \ell_1(\beta/2^{i+3})) \tag{12.34}$$

for $i \geq 0$. The parameter β is free for the moment. As for (12.28), (12.34) implies

$$J_2(U) \leq kc_2(\psi_0, \psi_1) + \beta/2. \tag{12.35}$$

Now we modify the proof of Theorem 9.9; see also [30, 31]. As in (12.28), for each $j, i \leq j \leq i + k - 1$,

$$J_2(U) \geq J_2(f_j(U)) + (k-1)c_2(\psi_0, \psi_1), \tag{12.36}$$

so combining (12.35)–(12.36) gives

$$J_2(f_j(U)) \leq c_2(\psi_0, \psi_1) + \beta/2. \tag{12.37}$$

Choose

$$\beta/2 < \overline{\delta}_2(\sigma), \tag{12.38}$$

where σ is free for now. Therefore by Proposition 12.22 and ($\mathcal{M}_2$), there is a $g_j \in \mathcal{M}_2(\psi_j, \psi_{j+1})$ such that

$$\|f_j(U) - g_j\|_{W^{1,2}(\cup_{\ell=-2}^{2} S_{q+\ell})} \leq \sigma \tag{12.39}$$

for all $q \in \mathbb{Z}$. But $U \in Z(i,k)$, so

$$\int_{T_{-j,m_j}} f_j(U)dx \in [s_0, t_0]. \tag{12.40}$$

By $(\mathcal{M}_2)$ and (12.4)–(12.5), we can further restrict s_0, t_0 so that

$$\int_{T_{0,0}} h_1 dx \equiv s_{-1} < s_0 < \int_{T_{0,0}} h_0\, dx < t_0 < \int_{T_{0,0}} h_{-1}\, dx \equiv t_1. \tag{12.41}$$

Now we impose our first restriction on σ:

$$0 < \sigma < \min(t_1 - t_0, s_0 - s_{-1}). \tag{12.42}$$

Let $g = \tau^2_{-m_j} g_j \in \mathcal{M}_2(\psi_0, \psi_1)$. Then

$$\begin{aligned}\int_{T_{-j,0}} g(x)dx = \int_{T_{-j,m_j}} g_j(x)dx \le \int_{T_{-j,m_j}} f_j(U)dx + \int_{T_{-j,m_j}} |f_j(U) - g_j|dx \\ < t_0 + \sigma < t_1\end{aligned} \tag{12.43}$$

via (12.39)–(12.41). Thus (12.43) combined with a similar lower bound yields

$$\int_{T_{-j,0}} g\, dx \in (s_{-1}, t_1). \tag{12.44}$$

There is a unique $h^* \in \mathcal{M}_2(\psi_j, \psi_{j+1})$ with

$$\int_{T_{-j,0}} h^*\, dx \in (s_{-1}, t_1), \tag{12.45}$$

namely $h^* = h_j$. Hence (12.44)–(12.45) show that

$$g_j = \tau^2_{m_j} h_j = \tau^2_{m_j} \tau^1_j h_0.$$

Now finally to verify (12.11) or equivalently

$$\psi_j < f_j(U) < \psi_{j+1} \;\; \text{on} \;\; \tau^1_{-j} \tau^2_{-m_j} A_{N,R}, \tag{12.46}$$

we modify (9.71)–(9.79). Choose $\theta = \theta(N, R)$ such that

$$0 < \theta < \frac{1}{4} \min_{A_{N+1,R+2}} (\psi_1 - h_0, h_0 - \psi_0). \tag{12.47}$$

Such a θ can be found, since $\psi_0 < h_0 < \psi_1$ on $\mathbb{R}^2 \times \mathbb{T}^{n-2}$. Since for any $j \in \mathbb{Z}$, $\psi_{j+1} - h_j = \tau^1_{-j}(\psi_1 - h_0)$ and $h_j - \psi_j = \tau^1_{-j}(h_0 - \psi_0)$, (12.47) implies

$$0 < \theta < \frac{1}{4} \min_{\tau^1_j \tau^2_{-m_j} A_{N+1,R+2}} (\psi_{j+1} - g_j, g_j - \psi_j). \tag{12.48}$$

To get the upper bound in (12.46), set $\varphi_j = \max(f_j(U) - g_j, 0)$. If the upper bound fails to hold for some j, there is a $q_j \in \tau^1_j \tau^2_{-m_j} A_{N-1,R+2} \equiv \mathcal{A}_j$ such that

$$\varphi_j(q_j) = f_j(U(q_j)) - g_j(q_j) = \psi_{j+1}(q_j) - g(q_j) \geq 4\theta. \tag{12.49}$$

Therefore there are $\xi, \eta \in \mathbb{R} \times Z$ such that

$$q_j \in [\xi, \xi + 1] \times [\eta, \eta + 1] \times \mathbb{T}^{n-2} \equiv D \subset \mathcal{A}_j.$$

By (12.39),

$$|\{\varphi_j > \theta\} \cap D|\theta^2 \leq \| f_j(U) - g_j\|_{L^2(D)} \leq \sigma^2. \tag{12.50}$$

Note that $v_0 < U, g_j < w_0$ and U and g_j are solutions of (PDE). Hence U and g_j are bounded in $C^2(\mathbb{R}^2 \times \mathbb{T}^{n-2})$, and as in (9.75), there is a constant M such that

$$\|\nabla U\|_{L^\infty(\mathbb{R}^2 \times \mathbb{T}^{n-2})}, \|g_j\|_{L^\infty(\mathbb{R}^2 \times \mathbb{T}^{n-2})} \leq M. \tag{12.51}$$

Set

$$r = \min\left(1, \frac{\theta}{2M}\right), \tag{12.52}$$

so $\overline{B}_r(q_j) \subset \cup^1_{\ell=-1} S_{\eta+\ell}$. Further requiring σ to satisfy (9.77) and then following (9.78)–(9.79) yields a contradiction to (12.52). The lower bound in (12.46) is obtained in a similar fashion, and the proof of Theorem 12.8 is complete.

Remark 12.53. (i) Since $g_i \leq \tau^2_{-1} g_i$,

$$0 < 4\theta < \psi_{i+1} - g_i \text{ on } [-N - i - 1, N - i + 1] \times (-\infty, m_i + R + 2) \times \mathbb{T}^{n-2},$$

and similarly

$$0 < 4\theta < g_{i+k-1} - v_{i+k-1}$$

on

$$[-N - (i + k - 1), N - (i + k - 1)] \times [m_{i+k-1} - R - 2, \infty) \times \mathbb{T}^{n-2}.$$

(ii) For later purposes, N will be chosen such that $\psi_0 - v_0 \geq (1 - \epsilon)(w_0 - v_0)$ on T_{N-1}, where ϵ is free for the moment.

Next we will prove an infinite-transition version of Theorem 12.8.

Theorem 12.54. *Under the hypotheses of Theorem 12.8:*

1º There is a solution $U^* = U^*_{R,N}$ of (PDE) satisfying (12.9) and also (12.6) and (12.11) for all $j \in \mathbb{Z}$.
2º There are solutions $\psi^{\pm} \in C^2(\mathbb{R} \times \mathbb{T}^{n-1})$, $\varphi^{\pm} \in C^2(\mathbb{T}^1 \times \mathbb{R} \times \mathbb{T}^{n-2})$ of (PDE) defined by

$$\psi^{\pm}(x) = \lim_{\ell \to \pm\infty} U^*(x + \ell e_2), \tag{12.55}$$

$$\varphi^{\pm}(x) = \lim_{\ell \to \pm\infty} U^*(x + \ell e_1), \tag{12.56}$$

convergence being in C^2_{loc} and

$$\begin{aligned} &v_0 \le \psi^- < U^* < \psi^+ \le w_0, \\ &v_0 \le \varphi^- < U^* < \varphi^+ \le w_0. \end{aligned} \tag{12.57}$$

3º There is a $\rho = \rho(N) \in (0,1)$ with $\rho(N) \to 0$ as $N \to \infty$ such that

$$\begin{cases} \|\psi^- - v_0\|_{L^\infty(\mathbb{R}\times\mathbb{T}^{n-1})}, \|\varphi^- - v_0\|_{L^\infty(\mathbb{T}^1\times\mathbb{R}\times\mathbb{T}^{n-2})} \\ \qquad \le \rho(N)\|w_0 - v_0\|_{L^\infty(T_0)}, \\ \|w_0 - \psi^+\|_{L^\infty(\mathbb{R}\times\mathbb{T}^{n-1})}, \|w_0 - \varphi^+\|_{L^\infty(\mathbb{T}^1\times\mathbb{R}\times\mathbb{T}^{n-2})} \\ \qquad \le \rho(N)\|w_0 - v_0\|_{L^\infty(T_0)}. \end{cases} \tag{12.58}$$

In fact more is true, and indeed this is the main result in this section:

Theorem 12.59. *Under the hypotheses of Theorem 12.54,*

$$\psi^- = v_0 = \varphi^-, \psi^+ = w_0 = \varphi^+.$$

Thus Theorem 12.59 gives us the existence of a solution of (PDE) that undergoes an infinite number of transitions and is heteroclinic in both x_1 and x_2 from v_0 to w_0. The proof of Theorem 12.54 is fairly straightforward, but some new ideas are required for that of Theorem 12.59. In particular, a monotone rearrangement argument will play a major role.

Proof of Theorem 12.54. Take $i = -q$ and $k = 2q + 1$ in Theorem 12.8 giving us a solution $U_q = U_{-q,2q+1} \in Z(-q, 2q + 1)$ of (PDE) satisfying (12.9) as well as (12.6) and (12.11) for $-q \le j \le q$. As in (12.51), for any $\alpha \in (0,1)$, there is an $M = M(\alpha)$ such that

$$\|U_q\|_{C^{2,\alpha}(\mathbb{R}^2\times\mathbb{T}^{n-2})} \le M. \tag{12.60}$$

Consequently, there is a $U^* \in C^{2,\alpha}(\mathbb{R}^2 \times \mathbb{T}^{n-2})$ such that $U_q \to U^*$ in $C^{2,\alpha}_{\text{loc}}(\mathbb{R}^2 \times \mathbb{T}^{n-2})$ along a subsequence. This convergence implies that U^* is a solution of (PDE) and satisfies (12.6) and (12.11) for all $j \in \mathbb{Z}$ as well as (12.9). By a familiar argument, (12.9) holds with strict inequality. Thus we have 1^o of Theorem 12.54.

Next for $\ell \in \mathbb{Z}$ and $x_2 \in [0, 1]$, define $u_\ell(x) = U^*(x + \ell e_2)$. By (12.9),

$$u_\ell(x) < u_{\ell+1}(x), \tag{12.61}$$

and u_ℓ is 1-monotone in x_1. The functions (u_ℓ) satisfy (PDE), and by (12.60),

$$\|u_\ell\|_{C^{2,\alpha}(\mathbb{R}\times[0,1]\times\mathbb{T}^{n-1})} \leq M.$$

Hence, as above, there are solutions $\psi^\pm \in C^{2,\alpha}(\mathbb{R} \times \mathbb{T}^{n-1})$ of (PDE) such that $u_\ell \to \psi^\pm$ as $\ell \to \pm\infty$. Note that by (12.61), the entire sequence converges and $\psi^\pm$ are 1-periodic in x_2. A similar analysis of $U^*(x + \ell e_1)$ yields 2^o of Theorem 12.54.

Now to obtain ρ and prove 3^o, observe that for any $j \in \mathbb{Z}$, $x \in [-N, N] \times [0, 1] \times \mathbb{T}^{n-2}$, and $q \in \mathbb{N}$ with $q \geq j$,

$$\begin{aligned} v_0(x) < \psi_0(x) &= \psi_j(x - je_1 + m_j e_2) \\ &\leq U_q(x - je_1 + m_j e_2) \leq \psi_0(x + e_1) < w_0(x). \end{aligned} \tag{12.62}$$

As $x_1 \to -\infty$, $\psi_0(x) - v_0(x) \to 0$, and as $x \to \infty$, $w_0(x) - \psi_0(x) \to 0$ uniformly for $(x_2, \ldots, x_n) \in [0, 1] \times \mathbb{T}^{n-2}$. Thus letting $q \to \infty$, (12.62) implies that there is a $\rho^-(N) \in (0, 1)$ with $\rho^-(N) \to 0$ as $N \to \infty$ and

$$U^*(x - je_1 + m_j e_2) - v_0(x) \leq \rho^-(N)\|w_0 - v_0\|_{L^\infty(T_0)} \tag{12.63}$$

for $x \in [-N, -N + 1] \times [0, 1] \times \mathbb{T}^{n-2}$. Since U^* is 1-monotone in x_2, for $\ell \leq m_j$, by (12.63),

$$U^*(x - je_1 + \ell e_2) - v_0(x) \leq \rho^-(N)\|w_0 - v_0\|_{L^\infty(T_0)}. \tag{12.64}$$

Thus letting $\ell \to -\infty$, (12.64) implies

$$\psi^-(x - je_1) - v_0(x) \leq \rho^-(N)\|w_0 - v_0\|_{L^\infty(T_0)}. \tag{12.65}$$

Since j is arbitrary,

$$\psi^-(x) - v_0(x) \leq \rho^-(N)\|w_0 - v_0\|_{L^\infty(T_0)} \tag{12.66}$$

for all $x \in \mathbb{R}^2 \times \mathbb{T}^{n-2}$.

Similarly, there is a $\rho^+(N) \in (0, 1)$ such that

$$w_0(x) - \psi^+(x) \leq \rho^+(N)\|w_0 - v_0\|_{L^\infty(T_0)} \tag{12.67}$$

for all $x \in \mathbb{R}^2 \times \mathbb{T}^{n-2}$. Thus we have obtained the part of (12.58) involving $\psi^{\pm}$. To get the analogous estimate for $\varphi^{\pm}$, in (12.63) write $x = y - Ne_1$, where $y \in [0,1]^2 \times \mathbb{T}^{n-2}$. Since U^* is 1-monotone in x_1, for $\ell \in \mathbb{N}$,

$$U^*(y - (\ell + j)e_1 + m_j e_2) - v_0(y) \leq \rho^-(N)\|w_0 - v_0\|_{L^\infty(T_0)}. \tag{12.68}$$

Letting $\ell \to \infty$,

$$\varphi^-(y + m_j e_2) - v_0(y) \leq \rho^-(N)\|w_0 - v_0\|_{L^\infty(T_0)}. \tag{12.69}$$

Now using the fact that (12.69) holds for all $j \in \mathbb{Z}$ and that φ^- is 1-monotone in x_2, (12.69) yields

$$\varphi^-(x) - v_0(x) \leq \rho^-(N)\|w_0 - v_0\|_{L^\infty(T_0)}, \tag{12.70}$$

and similarly

$$w_0(x) - \varphi^+(x) \leq \rho^+(N)\|w_0 - v_0\|_{L^\infty(T_0)}. \tag{12.71}$$

These estimates yield 3^o, and the proof of Theorem 12.54 is complete.

Next we will carry out the proof of Theorem 12.59 for ψ^+. The remaining cases are treated similarly. For what follows, the hypotheses of Theorem 12.8 will always be assumed. The key step in the proof is:

Proposition 12.72. *ψ^+ is minimal and is strictly 1-monotone in x_1.*

Assuming Proposition 12.72 for the moment, we immediately have the

Proof of Theorem 12.59. By Theorem 12.54, $\psi^+ \in C^2(\mathbb{R} \times \mathbb{T}^{n-1})$ and is a solution of (PDE). Since $\rho < 1$, by (12.56), $\psi^+ \notin \{v_0\} \cup \Gamma_1(v_0, w_0) \cup \Gamma_1(w_0, v_0)$. Hence by Proposition 12.72 and Theorem 3.60, $\psi^+ \equiv w_0$.

Now finally we give the

Proof of Proposition 12.72. By Theorem 12.8, $U^* < \tau_{-1}^1 U^*$ and by Theorem 12.54 for $x \in \mathbb{R} \times \mathbb{T}^{n-1}$, $U^*(x + \ell e_2) \to \psi^+(x)$ as $\ell \to \infty$ uniformly on compact sets. Therefore $\psi^+ \leq \tau_{-1}^1 \psi^+$, i.e., ψ^+ is 1-monotone in x_1. Moreover, by earlier maximum principle arguments, it is strictly 1-monotone in x_1.

To prove that ψ^+ is minimal, (1.1) must be verified. If it is false, there are a $\chi \in W^{1,2}(\mathbb{R}^n)$ having compact support and an $\alpha = \alpha(\chi) > 0$ such that

$$\int_{\mathbb{R}^n} (L(\psi^+ + \chi) - L(\psi^+))dx = \int_{\text{supp } \chi} (L(\psi^+ + \chi) - L(\psi^+))dx \leq -3\alpha, \tag{12.73}$$

where supp χ denotes the support of χ. Set $g_\ell = \tau_{-\ell}^2 U^*$. By (12.55) and (12.73), there is an $\ell_0 \in \mathbb{N}$ such that for $\ell \geq \ell_0$, $\ell \in \mathbb{N}$,

$$\int_{\text{supp } \chi} (L(g_\ell + \chi) - L(g_\ell))dx = \int_{\text{supp } \tau_\ell^2 \chi} (L(U^* + \tau_\ell^2 \chi) - L(U^*))dx \leq -2\alpha. \tag{12.74}$$

The proof of Theorem 12.54 gives U^* as the C^2_{loc} limit of a subsequence of $U_{-t,t-1} \equiv U_t$, i.e., there is a sequence $t_i \to \infty$ as $i \to \infty$ such that $U_{t_i} \equiv V_i \to U^*$ in C^2_{loc}. Therefore for each $\ell \geq \ell_0$, there is an $i = i_0(\ell)$ such that for $i \geq i_0(\ell)$,

$$\int_{\text{supp } \tau_\ell^2 \chi} (L(V_i + \tau_\ell^2 \chi) - L(V_i))dx \leq -\alpha. \tag{12.75}$$

We will show that (12.75) is not possible if $\ell = \ell(\chi)$ is sufficiently large.

Choose $r \in \mathbb{N}$ and $z \in \mathbb{Z}^2 \times \mathbb{T}^{n-2}$ such that supp $\chi \subset B_{\frac{r}{2}}(z)$. For any $\ell \geq \ell_0$, set $\tau_\ell^2 B_r(z) = B_r(z + \ell e_2) \equiv B^*$.

Note that

$$B^* \subset K_\ell \equiv [z_1 - r, z_1 + r] \times [z_2 + \ell - r, z_2 + \ell + r] \times \Pi_{i=3}^n [z_i - r, z_i + r].$$

Recall that $m_{k+1} - m_k \to \infty$ as $k \to \infty$. Thus for sufficiently large $k \in \mathbb{N}$, we can find a $t \geq \ell_0$ such that

$$-k + N < z_1 - r \tag{12.76}$$

and

$$m_k + 1 < z_2 + t - r < z_2 + t + r < m_{k+1} - 1. \tag{12.77}$$

Set $\ell = t$ and choose $i \geq i_0(\ell)$ such that $t_i > k$.

We will show that (12.75) cannot hold for such an i and ℓ. Toward that end, with ℓ and i now fixed, set

$$A_i = \{u \in W^{1,2}_{\text{loc}}(\mathbb{R}^2) \mid u = V_i \text{ in } \mathbb{R}^n \backslash B^*\}$$

and consider the minimization problem

$$\inf_{u \in A_i} \Phi(u), \tag{12.78}$$

where

$$\Phi(u) = \int_{B^*} L(u)dx.$$

Standard lower semicontinuity arguments show that there is a minimizer W of (12.78), in the closed convex set A_i. Moreover, since $V_i \in C^{2,\beta}(\mathbb{R}^n)$ for any $\beta \in (0,1)$, elliptic regularity results (see, e.g., [29]) imply $W \in C^{2,\beta}(B^*)$.

Since supp $\tau_\ell^2\chi \subset B^*$, $V_i + \tau_\ell^2\chi \in A_i$. Therefore by (12.78) and (12.75),

$$\Phi(W) \le \Phi(V_i + \tau_\ell^2\chi) \le \Phi(V_i) - \alpha. \tag{12.79}$$

Using the minimality properties of the functions ψ_j and arguing as in the proof of Theorem 3.2 or Theorem 9.6 shows that

$$\psi_{-t_i} \le W \le \psi_{t_i}. \tag{12.80}$$

By (12.77), W satisfies (12.6) for $-t_i \le j \le t_i - 1$. Thus W satisfies two of the four requirements for membership in $Z(-t_i, 2t_i)$. If $W \in Z(-t_i, 2t_i)$,

$$\Phi(V_i) \le \Phi(W), \tag{12.81}$$

contrary to (12.79), and Proposition 12.72 is proved. Unfortunately, a priori W need not be 1-monotone in x_1 or x_2. However, we will show that there is a rearrangement W_2 of W for which $J_2(W_2) = J_2(W)$ and $W_2 \in Z(-t_i, 2t_i)$. Thus the above argument holds with W replaced by W_2.

To obtain W_2, first some estimates are needed for W. By Theorem 12.8 and (ii) of Remark 12.53, for $\overline{x} \in [-k+N-1, -k+N] \times [m_k, m_k+1] \times \mathbb{T}^{n-2}$,

$$w_0(\overline{x}) - \epsilon(w_0(\overline{x}) - v_0(\overline{x})) \le \psi_k(\overline{x}) = \psi_0(\overline{x} + ke_1) < V_i(\overline{x}).$$

Since V_i is 1-monotone in x_1 and x_2, for $s, t \in \mathbb{N}$,

$$w_0(\overline{x}) - \epsilon(w_0(\overline{x}) - v_0(\overline{x})) < V_i(\overline{x} + se_1 + te_2). \tag{12.82}$$

In particular, by (12.76)–(12.77), (12.82), on K_ℓ,

$$0 < w_0 - V_i < \epsilon(w_0 - v_0). \tag{12.83}$$

In fact, (12.83) holds for $x_1 \ge -k + N - 1$ and $x_2 \ge m_k$ and also on $\hat{K}$, a unit neighborhood of K_ℓ. Thus by the interior L^p elliptic theory, see, e.g., Section 9.5 of Gilbarg–Trudinger [29] and (12.83), for any $p > 1$,

$$\begin{aligned} \|w_0 - V_i\|_{W^{2,p}(K_\ell)} &\le a_1(\|\Delta(w_0 - V_i)\|_{L^p(\hat{K})} + \|w_0 - V_i\|_{L^p(\hat{K})}) \\ &\le a_1(\|F_u(\cdot, w_0) - F_u(\cdot, V_i)\|_{L^p(\hat{K})} + \epsilon\|(w_0 - v_0)\|_{L^\infty}|\hat{K}|^{1/p}), \end{aligned} \tag{12.84}$$

where $|\hat{K}|$ denotes the measure of $\hat{K}$ and a_1 depends on p and r but not i. For $p > n$, by the Sobolev inequality,

$$\|w_0 - V_i\|_{C^1(K_\ell)} \le a_2\|w_0 - V_i\|_{W^{2,p}(K_\ell)}, \tag{12.85}$$

where a_2 depends on p and r but not i. Observe that by (12.83)–(12.85),

$$V_i \to w_0 \text{ in } C^1(K_\ell) \text{ as } \epsilon \to 0. \tag{12.86}$$

Therefore

$$\Phi_\ell(V_i) \equiv \int_{K_\ell} L(V_i)dx \to \Phi_\ell(w_0) \text{ as } \epsilon \to 0. \tag{12.87}$$

Let

$$\overline{W}_1 = \begin{cases} w_0, & \mathbb{R}^n \backslash K_\ell, \\ W, & K_\ell \backslash D_1, \end{cases}$$

where

$$D_1 = (\{z_1 - r \le x_1 \le z_1 - r + 1\} \cup \{z_1 + r - 1 \le x_1 \le z_1 + r\}),$$

and interpolate in D_1 in the usual way. Define $\overline{W}_2$ analogously with

$$D_2 = (\{z_2 + \ell - r \le x_2 \le z_2 + \ell - r + 1\} \cup \{z_2 + \ell + r - 1 \le x_2 \le z_2 + \ell + r\}).$$

Thus we have

$$\|\overline{W}_2 - W\|_{W^{1,2}(K_\ell)} \le \overline{c}\|W - w_0\|_{W^{1,2}((D_1 \cup D_2) \cap K_\ell)} \to 0$$

as $\epsilon \to 0$, since $W = V_i$ on $K_\ell \backslash B^*$. Thus

$$|\Phi_\ell(\overline{W}_2) - \Phi_\ell(W)| \to 0,$$

which combined with (12.79) and $V_i \to w_0$ implies $\Phi_\ell(\overline{W}_2) \to \Phi_\ell(w_0)$. Given that w_0 is a minimizer of Φ_ℓ over $2r$-periodic functions in $W^{1,2}(K_\ell)$, and $\overline{W}_2$ can be extended to such a function, we have $\|\nabla(\overline{W}_2 - w_0)\|_{L^p(K_\ell)} \to 0$ which combined with (12.86) yields $\|\nabla(W - V_i)\|_{L^p(K_\ell)} \to 0$, as $\epsilon \to 0$.

By the Poincaré inequality,

$$\|W - V_i\|_{L^2(B^*)} \le a_3\|\nabla(W - V_i)\|_{L^2(B^*)}, \tag{12.88}$$

where a_3 depends on r. Therefore

$$\|W - V_i\|_{W^{1,2}(B^*)} \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{12.89}$$

Finally, by the Sobolev inequality again with $p > n$ as above, we have

$$\begin{aligned} \|W - V_i\|^p_{L^\infty(B^*)} &\le a_4\|W - V_i\|^p_{W^{1,p}(B^*)} \\ &\le a_4\|W - V_i\|^{p-2}_{C^1(B^*)}\|W - V_i\|^2_{W^{1,2}(B^*)}. \end{aligned} \tag{12.90}$$

Since $v_0 < W, V_i < w_0$ on B^*, both satisfy (PDE) on B^*, and F_u is bounded on $C^1(\mathbb{R}^n \times [0, 1])$, standard elliptic estimates show that $\|W\|_{C^2(B^*)}$, $\|V_i\|_{C^2(B^*)}$ are bounded independently of i and ℓ. Therefore by (12.89)–(12.90), for ϵ small enough (or N large enough), it can be assumed that

$$w_0 - W \leq (w_0 - v_0)/3 \ \text{ on } \ B^* \tag{12.91}$$

and by (12.88),

$$w_0 - V_i(x) \leq (w_0 - v_0)/3, \ \ x_1 \geq -k + N - 1, x_2 \geq m_k. \tag{12.92}$$

Next, to determine W_2, note first that since $W = V_i$ outside of B^* and V_i satisfies (12.80) with strict inequality,

$$\psi_{-n_i} < W < \psi_{n_i}, \ \text{ on } \ \mathbb{R}^n \backslash B^*. \tag{12.93}$$

We claim that (12.93) also holds in B^*. This follows from the maximum principle argument of, e.g., Proposition 2.2.

The function W_2 will be constructed by rearranging W on a region of the form $\mathcal{R} = [p_i, q_i] \times [P_i, Q_i] \times \mathbb{T}^{n-2}$. To determine the parameters p_i, etc., recall that $-\psi_{t_i} < V_i = W < \psi_{t_i}$ on $\mathbb{R}^n \backslash B^*$, $\psi_0(x) \to v_0$ as $x_1 \to -\infty$, and $\psi_0(x) \to w_0$ as $x_1 \to \infty$ uniformly in x_2. Choose p_i such that

$$\psi_{t_i}|_{T_{p_i}} - v_0 < (w_0 - v_0)/3 \tag{12.94}$$

and choose q_i such that

$$\psi_{-t_i}|_{T_{q_i - 1}} - v_0 > \left(\max_{K_\ell} \frac{W - v_0}{w_0 - v_0}\right)(w_0 - v_0). \tag{12.95}$$

With p_i and q_i now fixed, note that

$$\|W - \psi_{n_i}\|_{W^{1,2}(S_j)} \to 0, \ \ j \to \infty. \tag{12.96}$$

Using (PDE) and interpolation arguments, (12.96) implies

$$\|W - \psi_{n_i}\|_{L^\infty(S_j)} \to 0, \ \ j \to \infty. \tag{12.97}$$

Consequently, there is a $Q_i \in \mathbb{N}$, $Q_i \geq m_{n_i} + 1$, such that for all x with $x_1 \in [p_i, q_i]$ and $x_2 \in [z_2 + \ell - r, z_2 + \ell + r]$,

$$W(x) < W(\tilde{x}), \ \ \tilde{x} \in S_j \cap O_2^+(x), j \geq Q_i - 1, \tag{12.98}$$

where for $i \in \{1, 2, \ldots, n\}$,

$$O_i(x) = \{x + \ell e_i \mid \ell \in \mathbb{Z}\}, \ O_i^{\pm}(x) = \{x \pm \ell e_i \mid \ell \in \mathbb{N}\}. \tag{12.99}$$

Similarly, there is a $P_i \in -\mathbb{N}$, $P_i \leq m_{-n_i} - 1$ such that for all x with $x_1 \in [p_i, q_i]$ and $x_2 \in [z_2 + \ell - r, z_2 + \ell + r]$,

$$W(\hat{x}) < W(x), \quad \hat{x} \in S_j \cap O_2^-(x), \quad j \leq P_i + 1. \tag{12.100}$$

Next we define our rearrangement and establish some of its basic properties. Let $a, b, c, d, \alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $a + 1 < b < c < d$, $\alpha < \beta < \gamma < \delta$. Consider $u \in C(\mathbb{R}^n) \cap W^{1,2}_{\text{loc}}(\mathbb{R}^n)$ such that

$$u \text{ is 1-monotone in } x_2 \text{ for } (x_1, x_2) \notin (\beta, \gamma) \times (b, c) \tag{12.101}$$

and satisfies

$$u(\hat{x}) \leq u(x) \leq u(\bar{x}) \tag{12.102}$$

for all $(x_1, x_2) \in [\beta, \gamma] \times [b, c]$ with $\hat{x} \in O_2^-(x), \bar{x} \in O_2^+(x)$ uniquely determined by $\hat{x}_2 \in (a, a + 1]$, and $\bar{x}_2 \in (d - 1, d]$.

The rearranged function (with respect to x_2) will be denoted by $R_{x_2}u(x)$. Define $R_{x_2}u(x) = u(x)$ for $(x_1, x_2) \notin (\alpha, \delta) \times (a, d]$; for $\alpha < x_1 < \delta$, $a < x_2 \leq d$, take $j \in \mathbb{N}$ such that $a + j - 1 < x_2 \leq a + j$ and let $R_{x_2}u(x)$ be the jth smallest of the numbers $u(\tilde{x})$, $\tilde{x} \in O_2(x)$, $a < \tilde{x}_2 \leq d$.

Proposition 12.103. *If u is in $C(\mathbb{R}^n) \cap W^{1,2}_{\text{loc}}(\mathbb{R}^n)$ and satisfies (12.101)–(12.102), then $R_{x_2}u(x)$ is in $C(\mathbb{R}^n) \cap W^{1,2}_{\text{loc}}(\mathbb{R}^n)$ and is 1-monotone in x_2 and $J_2(R_{x_2}u) = J_2(u)$. In addition,*

$$R_{x_2}u(x) \leq u(x) \text{ for } x_2 \leq b, \quad R_{x_2}u(x) \geq u(x) \text{ for } x_2 \geq c, \tag{12.104}$$

while $R_{x_2}u(x) = u(x)$

$$\text{if } x_2 \leq b \text{ and } R_{x_2}u(x) \leq \min_{x^* \in O_2^+(x), x_2^* \in [b,c]} u(x^*), \tag{12.105}$$

$$\text{if } x_2 \geq c \text{ and } R_{x_2}u(x) \geq \max_{x^* \in O_2^-(x), x_2^* \in [b,c]} u(x^*), \tag{12.106}$$

and on

$$\mathbb{R}^n \backslash [(\beta, \gamma) \times (a + 1, d - 1]]. \tag{12.107}$$

Also if $\hat{v}$ is 1-monotone in x_2,

$$u(x) \leq \hat{v}(x) \text{ for } x_2 \leq \mu \Rightarrow R_{x_2}u(x) \leq \hat{v}(x) \text{ for } x_2 \leq \mu \tag{12.108}$$

and

$$u(x) \geq \hat{v}(x) \text{ for } x_2 \geq \mu \Rightarrow R_{x_2}u(x) \geq \hat{v}(x) \text{ for } x_2 \geq \mu. \tag{12.109}$$

Finally,

$$\text{if } u \text{ is 1-monotone in } x_1, \text{ then } R_{x_2}u \text{ is as well.} \tag{12.110}$$

Remark 12.111. More generally, for fixed $x_i, i \neq 2$, if $u(x)$ is 1-monotone in x_2 for $x_2 \notin [b', c']$, then (12.104)–(12.106) hold with b, c replaced by b', c'.

Remark 12.112. When (12.105) holds, then from the proof of $R_{x_2}u(x) = u(x)$ in the case that (12.105) holds, we see that there are $\bar{x}, \tilde{x} \in O_2(x)$ with $a \leq \bar{x}_2 \leq x_2$, $b \leq \tilde{x}_2 \leq c$, and $R_{x_2}u(\bar{x}) = u(\tilde{x})$.

Proof of Proposition 12.103. By definition, $R_{x_2}u$ is 1-monotone in x_2. Next we will prove (12.104)–(12.110). For x_1, x_2, $a < x_1 < \delta$, and $a < x_2 \leq b$, let j be as in the definition of $R_{x_2}u$. By (12.101), $u(x) \geq u(x + (1-i)e_2), i = 1, 2, \ldots, j$, i.e., $u(x)$ bounds at least j of $u(\tilde{x})$, $\tilde{x} \in O_2$, $a < \tilde{x} \leq d$, and the first part of (12.104) is verified. The second part follows in a similar manner, since the jth smallest of $u(\tilde{x})$, $\tilde{x} \in O_2$, $a < \tilde{x}_2 \leq d$, is the $(\bar{q} + 1 - j)$th largest, where $\bar{q} = d - a$.

Again take x_1, x_2, $\alpha < x_1 < \beta$, $a < x_2 \leq b$, and j as in the definition of $R_{x_2}u$. Given $R_{x_2}u(x) \leq \min_{x^* \in O_2(x), x_2^* \in [b,c]} u(x^*)$, assume $R_{x_2}u(x) < u(x)$, so $u(x)$ strictly exceeds the j smallest values of the form $u(\tilde{x})$, $\tilde{x} \in O_2(x)$, $a < \tilde{x}_2 \leq d$. Thus these values contain $u(\tilde{x})$ for some $\tilde{x}_2 > x_2$. Note that by the 1-monotonicity of $u(x)$ in x_2 for $x_2 \notin (b, c)$ we have $\tilde{x}_2 \in (b, c)$. Also $R_{x_2}u(\overline{x}) = u(\tilde{x})$ is the ith smallest of such values for $i \leq j$, so $\bar{x}_2 \leq x_2$. Therefore for $x' \in O_2(x)$ with $x_2' = b, u(x') \geq u(x) > R_{x_2}u(x) \geq R_{x_2}u(\overline{x}) = u(\tilde{x}) \geq \min_{x^* \in O_2(x), x_2^* \in [b,c]} u(x^*)$, a contradiction. Hence $R_{x_2}u(x) \geq u(x)$ and (12.105) follows from (12.104). Property (12.106) follows in a similar manner by again considering largest values.

Given $x, \hat{x}, \bar{x}$ as in (12.102), note that (12.102) implies that $u(\hat{x}), u(\bar{x})$ are the smallest and largest respectively of $u(\tilde{x})$, $\tilde{x} \in O_2(x)$, $a < \tilde{x}_2 \leq d$, so $R_{x_2}u(x) = u(x)$ for $x \in \{\hat{x}, \bar{x}\}$. This combined with the definition of $R_{x_2}u$ exterior to $(\alpha, \delta) \times (a, d]$ then implies $R_{x_2}u(x) = u(x)$ for $x_2 \leq a + 1$ and $x_2 > d - 1$. Since u is 1-monotone in x_2 for $x_1 \leq \beta$ and $x_1 \geq \gamma$, we see that $R_{x_2}u(x) = u(x)$ for such x_1 as well, so (12.107) is verified.

To verify (12.108), assume $u(x) \leq \hat{v}(x)$ for $x_2 \leq \mu$. We need only consider $\alpha < x_1 < \delta$, since otherwise $R_{x_2}u = u$. Assume $x_2 \leq \mu$. The $x_2 \leq a$ case is again trivial, and the $x_2 > d$ case can easily be reduced to the $a < x_2 \leq d$ case since $\max_{x^* \in O_2(x), x_2^* \in [a,d]} u(x^*) = \max_{x^* \in O_2(x), x_2^* \in [a,d]} R_{x_2}u(x^*)$ due to (12.102). Therefore we assume $\alpha < x_1 < \delta$, $a < x_2 \leq d$. Take $j \in \mathbb{N}$ such that $a + j - 1 < x_2 \leq a + j$, so $a \leq a + j - i < x_2 + 1 - i \leq a + 1 + j - i \leq d$ for $i = 1, \ldots, j$. Also $x_2 + 1 - i \leq x_2 \leq \mu$ for such i, so $u(x + (1-i)e_2) \leq \hat{v}(x)$, since $\hat{v}$ is 1-periodic in x_2. Consequently, the j smallest of $u(\tilde{x})$, $\tilde{x} \in O_2(x)$, $a < \tilde{x}_2 \leq d$ are also bounded by $\hat{v}(x)$, so $R_{x_2}u(x) \leq \hat{v}(x)$, and (12.108) is verified. Property (12.109) follows in an analogous manner again using the fact that the jth smallest value is the $(\bar{q} + 1 - j)$th largest value.

To check (12.110), we again need only consider x_1, x_2 with $\alpha < x_1 < \beta$, $a < x_2 \leq d$, and take $j \in \mathbb{N}$ such that $a + j - 1 < x_2 \leq a + j$. Let $u(x^i + e_1)$, $i = 1, 2, \ldots, j$, be the j smallest values of the form $u(\tilde{x} + e_1)$, $\tilde{x} \in O_2(x)$,

$a < \tilde{x}_2 \le d$. Thus $R_{x_2}u(x + e_1) \ge u(x^i + e_i) \ge u(x^i)$, $i = 1, 2, \ldots, j$, via the 1-monotonicity of u in x_1. Therefore $R_{x_2}u(x + e_1) \ge R_{x_2}u(x)$.

Next define $\mathcal{R}_j = (\alpha, \delta) \times (a + j - 1, a + j], 1 \le j \le \bar{q} = d - a$. For $x \in \mathcal{R}_1$, set $\varphi_j(x) = u(x + (j-1)e_2)$, so φ_j is continuous in the closure of $\mathcal{R}_1$ and $\varphi_j \in W^{1,2}_{\text{loc}}(\mathcal{R}_1)$ for $1 \le j \le \bar{q}$. In addition,

$$\int_{\mathcal{R}} L(u)dx = \sum_{j=1}^{\bar{q}} \int_{R_1} L(\varphi_j)dx. \tag{12.113}$$

Now define $\bar{q}$ new functions on $\mathcal{R}_1$ as follows:

$$\begin{aligned} s_1(x) &= \min_{1 \le t \le \bar{q}} \varphi_t(x), \\ s_2(x) &= \text{2nd smallest of } \{\varphi_1(x), \ldots, \varphi_{\bar{q}}(x)\}, \\ &\cdots \\ s_{\bar{q}}(x) &= \max_{1 \le t \le \bar{q}} \varphi_t(x). \end{aligned} \tag{12.114}$$

Each of the functions s_j can be expressed iteratively as the maximum or minimum of a pair of $W^{1,2}(\mathcal{R}_1)$ functions. For example, let $f_1 = \min(\varphi_1, \varphi_2)$, $g_1 = \max(\varphi_1, \varphi_2)$ and $f_{i+1} = \min(f_i, \varphi_{i+2})$, $g_{i+1} = \max(f_i, \varphi_{i+2})$, $i = 1, \ldots, \bar{q} - 2$. Then $s_1 = f_{\bar{q}-1}$. Apply the same process to the functions g_i in place of φ_i in order to construct s_2, etc. This implies $s_j \in W^{1,2}(\mathcal{R}_1)$ but also, along with (12.113), that

$$\int_{\mathcal{R}} L(u)dx = \sum_{j=1}^{\bar{q}} \int_{\mathcal{R}_1} L(s_j)dx. \tag{12.115}$$

Note that

$$R_{x_2}u(x) = s_j(x - (j-1)e_2), x \in \mathcal{R}_j, \tag{12.116}$$

for $1 \le j \le \bar{q}$. Thus $J_2(u) = J_2(R_{x_2}u)$ by (12.115) and $R_{x_2}u(x) \in C(\mathcal{R}_j) \cap W^{1,2}(\mathcal{R}_j)$, $j = 1, \ldots, \bar{q}$. Take $\theta \in (0, 1)$ and replace a, d by $a - \theta, d + 1 - \theta$ and rearrange as before to produce $R^{\theta}_{x_2}u \in C(\mathbb{R}^n) \cap W^{1,2}(\mathcal{R}_j)$, $j = 1, \ldots, \bar{q}$. Observe that $R^{\theta}_{x_2}u = R_{x_2}u$, so in combination with (12.107) we see that $R_{x_2}u(x) \in C(\mathbb{R}^n) \cap W^{1,2}_{\text{loc}}(\mathbb{R}^n)$ and the proof of Proposition 12.103 is complete.

Now to complete the proof of Proposition 12.72, it suffices to verify:

Proposition 12.117. *There is a $W_2 \in Z(-n_i, 2n_i)$ such that $W_2 = W$ on $\mathbb{R}^n \backslash \mathcal{R}$ and $J_2(W_2) = J_2(W)$.*

Proof. For convenience, we can take $p_i = 0$, $P_i = 0$ and set $q_i = q$, $Q_i = Q$. As a first step toward obtaining W_2, let $W_1 = R_{x_2}W$ using $\alpha = 0, \beta = z_1 - r$, $\gamma = z_1 + r, \delta = q$ and $a = 0, b = z_2 + \ell - r, c = z_2 + \ell + r, d = Q, \bar{q} = Q$.

We claim that Proposition 12.103 applies to W, so $W_1 \in C(\mathbb{R}^n) \cap W^{1,2}_{\text{loc}}(\mathbb{R}^n)$, W_1 is 1-monotone in x_2, $J(W_1) = J(W)$,

$$W_1 \le W \ \text{ for } \ x_2 \le z_2 + \ell - r, \quad W_1 \ge W \ \text{ for } \ x_2 \ge z_2 + \ell + r, \tag{12.118}$$

while $W_1(x) = W(x)$

$$\text{if } \ x_2 \le z_2 + \ell - r \ \text{ and } \ W_1(x) \le \min_{x^* \in O_2(x), x_2^* \in [z_2+\ell-r, z_2+\ell+r]} W(x^*), \tag{12.119}$$

$$W_1 = W \ \text{ on } \mathbb{R}^n \setminus [(z_1 - r, z_1 + r) \times (1, Q-1) \times \mathbb{T}^{n-2}]. \tag{12.120}$$

Also if $\hat{v}$ is 1-monotone in x_2,

$$W \ge \hat{v} \ \text{ for } \ x_2 \ge \mu \Rightarrow W_1 \ge \hat{v} \ \text{ for } \ x_2 \ge \mu, \tag{12.121}$$

as well as the analogue with all inequalities reversed, and

$$-\psi_{n_i} \le W_1 \le \psi_{n_i}, \tag{12.122}$$

since $-\psi_{n_i} \le W \le \psi_{n_i}$ by (12.80).

To verify the claim, recall that $W = V_i$ in $\mathbb{R}^n \setminus B^*$, $B^* \subset (z_1 - r, z_1 + r) \times (z_2 + \ell - r, z_2 + \ell + r) \times \prod_{i=2}^n (z_i - r, z_i + r)$, the interior of K_ℓ. Thus basic boundary regularity results imply $W \in C(\mathbb{R}^2 \times \mathbb{T}^{n-2})$, since V_1 is smooth, and by construction $W \in W^{1,2}_{\text{loc}}(\mathbb{R}^2 \times \mathbb{T}^{n-2})$. Also, by Theorem 12.8, V_i is 1-monotone in x_1 and x_2, so (12.101) holds. Lastly, (12.102) follows from (12.98)–(12.100).

Now define $W_2 = R_{x_1} W_1$, where $R_{x_1} u$ is the analogue of $R_{x_2} u$ with the roles of x_1, x_2 reversed, taking $\alpha = 0, \beta = z_1 - r, \gamma = z_1 + r, \delta = q$ and $a = 0$, $b = 1, c = Q - 1, d = Q, \bar{q} = q$. We now claim the R_{x_1} version of Proposition 12.103 applies to W_1, in which case $J_2(W_2) = J_2(W_1) = J(W)$, $W_2 \in C(\mathbb{R}^2 \times \mathbb{T}^{n-2}) \cap W^{1,2}_{\text{loc}}(\mathbb{R}^2 \times \mathbb{T}^{n-2})$, $\psi_{-n_i} \le W_2 \le \psi_{n_i}$ due to (12.108), (12.109), and (12.122). Also W_2 is 1-monotone in x_1 by construction and 1-monotone in x_2 by (12.110). Thus Proposition 12.117 is established once we verify the claim and show that W_2 satisfies the constraints (12.6).

To verify the new claim, recall that $W_1 \in C(\mathbb{R}^n) \cap W^{1,2}_{\text{loc}}(\mathbb{R}^n)$, and note that (12.120) implies the R_{x_1} version of (12.101) for W_1 since W is 1-monotone in x_1 on $\mathbb{R}^2 \times \mathbb{T}^{n-2} \setminus [(z_1 - r, z_1 + r) \times (1, Q-1)] \times \mathbb{T}^{n-2}$. It remains to confirm the R_{x_1} version of (12.102) for $u = W_1$, i.e.,

$$W_1(\hat{x}) \le W_1(x) \le W_1(\bar{x}) \tag{12.123}$$

for all $x \in [z_1 - r, z_1 + r] \times [1, Q-1] \times \mathbb{T}^{n-2}$ with $\hat{x}_1 \in (0, 1]$, $\hat{x} \in O_1^-(x)$, and $\bar{x} \in (q-1, q]$, $\bar{x} \in O_1^+(x)$. Note that (12.91)–(12.92) imply $W \ge w_0 - (w_0 - v_0)/3$ on $[z_1 - r, \infty) \times [z_2 + \ell - r, \infty)$, so

$$W_1 \ge w_0 - (w_0 - v_0)/3 \text{ for } (x_1, x_2) \in [z_1 - r, \infty) \times [z_2 + \ell - r, \infty) \tag{12.124}$$

by (12.121). However, (12.80), (12.94), (12.120) imply $W_1 = W < v_0 + (w_0 - v_0)/3 = w_0 - 2(w_0 - v_0)/3$ for $x_1 \leq 1$, so the first inequality in (12.123) is verified for $x_2 \geq z_2 + \ell - r$.

Assume $W_1(\hat{x}) > W_1(x)$ for some $(x_1, x_2) \in [z_1 - r, z_1 + r] \times [1, z_2 + \ell - r)$ with $\hat{x} \in (0, 1]$, $\hat{x} \in O_1^-(x)$. Then as above, $W_1(x) < W_1(\hat{x}) < w_0 - 2(w_0 - v_0)/3$, and (12.119) and (12.124) imply $W_1(x) = W(x)$. Thus $W_1(x) = W(x) \geq W(\hat{x}) = W_1(\hat{x}) > W_1(x)$, a contradiction, and the first inequality in (12.123) follows. The second inequality in (12.123) is proved in an analogous manner, with (12.95) replacing (12.93) and the version of (12.121) with inequalities reversed is used instead of (12.121).

The last step in the proof is to verify that W_2 satisfies the constraints (12.6) on $X_j = [-j, -j+1] \times [m_j, m_j + 1] \times \mathbb{T}^{n-2}$, $-n_i \leq j \leq n_i$. First we verify that W_1 satisfies these constraints. We can assume $-j \in [z_1 - r, z_1 + r - 1]$, since $W_1 = W$ in X_j otherwise. For such j the sets X_j lie below K_ℓ due to (12.25). On X_j, $W = V_i$, and by Theorem 12.8, $\psi_j < V_i < \psi_{j+1}$. Since $\psi_{j+1}|_{X_j} = \psi_0|_{T_0}$, by the normalization (12.1), $W - v_0 \leq (w_0 - v_0)/3$ on X_j. On the other hand, from above (12.124), $W - v_0 \geq 2(w_0 - v_0)/3$ on K_ℓ. This implies $W_1 = W$ on X_j due to (12.119).

Next note that (12.118) and the 1-monotonicity of W in x_1 for $x_2 \geq z_2 + \ell + r$, (12.120), and the R_{x_1} version of (12.104) imply for $x_2 \geq z_2 + \ell + r$, $x_1 \leq z_1 - r$ that

$$\begin{aligned}\min_{x^* \in O_1(x), x_1^* \in [z_1 - r, \infty)} W_1(x^*) &\geq \min_{x^* \in O_1(x), x_1^* \in [z_1 - r, \infty)} W(\cdot, x_2, \dots) \\ &\geq W(x) = W_1(x) \geq W_2(x).\end{aligned}$$

Thus the R_{x_1} version of (12.105) implies $W_2 = W_1$ for $x_1 \leq z_1 - r$, $x_2 \geq z_2 + \ell + r$. Similarly, $W_2 = W_1$ for $x_1 \geq z_1 + r$, $x_2 \leq z_2 + \ell - r$. Therefore (12.6) holds in these regions, since $W_1 = W$ there.

The remaining constraint regions are contained in $(-\infty, z_1 + r) \times (-\infty, z_2 + \ell - r) \times \mathbb{T}^{n-2}$, since $m_k < z_2 + \ell - r < z_2 + \ell + r < m_{k+1}$. The 1-monotonicity of W in x_1 in this region and $W - v_0 \leq (w_0 - v_0)/3$ on X_j imply $W - v_0 \leq (w_0 - v_0)/3$ on $\bar{X}_j = (-\infty, -j+1] \times (-\infty, m_j + 1] \times \mathbb{T}^{n-2}$. For $x \in \bar{X}_j$, if $x_1 \leq z_2 - r$, then $W_1 = W$. If $x_1 \geq z_2 - r$, then the same is true, as in the proof above for X_j, so $W_1 = W$ on $\bar{X}_j$. Therefore W_1 is 1-monotone in x_1 on $\hat{X}_j = (-\infty, b_j'] \times (-\infty, m_j + 1]$ for $b_j' = \max(-j+1, z_1 - r)$. Considering Remark 12.111 with $b' = b_j'$ in the R_{x_1} version of Proposition 12.103, we have $W_2 \leq W_1$ on $\hat{X}_j$ by (12.104). If $W_2(x) < W_1(x)$ for $x \in X_j$, then by the R_{x_1} version of Remark 12.112 there are $\bar{x}, \tilde{x} \in O_1(x)$, $\bar{x}_1 \leq x_1$, $b_j' \leq \tilde{x}_1 \leq z_2 + r$ with $W_2(\bar{x}) = W_1(\tilde{x})$. By the 1-monotonicity of W_2 in x_1, it follows that

$$W_1(\tilde{x}) = W_2(\bar{x}) \leq W_2(x) < W_1(x) = W(x) \leq w_0 - 2(w_0 - v_0)/3.$$

Consequently $W_1(\tilde{x}) = W(\tilde{x}) \geq W(x)$ by the R_{x_1} version of (12.105) (since $z_1 - r \leq \tilde{x}_1 \leq z_1 + r$) and the 1-monotonicity of W in x_1, which contradicts the previous line, and the proof is complete.

Remark 12.125. Theorem 12.59 gives solutions of (PDE) that undergo an infinite number of transitions in x_2 and are heteroclinic from v_0 to w_0. The same argument gives heteroclinics in x_2 from ψ_j to w_0 and from v_0 to ψ_k for any $j, k \in \mathbb{Z}$. Suppose that in analogy to $(\mathcal{M}_1)$,

$$\mathcal{M}_1(w_0, v_0) = \{\tau^1_{-j}\bar{v}_1 \equiv \chi_j \,|\, j \in \mathbb{Z}\},$$

where $\bar{v}_1 \in \mathcal{M}_1(w_0, v_0)$. Then

$$\mathcal{M}_1(1 + w_0, 1 + v_0) = \{1 + \tau^1_{-j}\bar{v}_1 \equiv 1 + \chi_j \,|\, j \in \mathbb{Z}\}$$

and we can seek solutions of (PDE) that are heteroclinic in x_2 from ψ_j to $1 + \chi_k$ or from χ_k to $1 + \psi_j$. Whether such solutions exist remains an open question.

Chapter 13
Solutions of (PDE) with Two Transitions in x_1 and Heteroclinic Behavior in x_2

The goal of this chapter is to construct another class of solutions of (PDE) that belong to $C^2(\mathbb{R}^2 \times \mathbb{T}^{n-2})$. As $x_2 \to \pm\infty$, the solutions we seek approach two transition solutions of the type considered in Chapter 9.

We first introduce the solutions to be used as asymptotic limits. As in Chapter 9, assume $v_0, w_0, \widehat{v}_0, \widehat{w}_0 \in \mathcal{M}_0$, where $v_0 < w_0 \le \widehat{v}_0 < \widehat{w}_0$ and the pairs v_0, w_0 and $\widehat{v}_0, \widehat{w}_0$ satisfy $(*)_0$. Suppose also there exist $v_1, w_1 \in \mathcal{M}_1(v_0, w_0)$ and $\widehat{v}_1, \widehat{w}_1 \in \mathcal{M}_1(\widehat{v}_0, \widehat{w}_0)$, where $v_1 < w_1 < \widehat{v}_1 < \widehat{w}_1$ and the pairs v_1, w_1 and $\widehat{v}_1, \widehat{w}_1$ satisfy $(*)_1$.

Define $\mathcal{C}_0^i, i = 1, 2,$ as following Remark 9.93 but with T_0 replaced by $B_{1/4}(p_0) = \{x \mid |x - p_0| < 1/4\}$, with p_0 the center of T_0. Likewise, as preceding Proposition 9.107, choose s_i, t_i such that $v_1 \in \mathcal{C}_0^1$, $w_1 \notin \mathcal{C}_0^1$, $v_1 \notin \mathcal{C}_0^2$, $w_1 \in \mathcal{C}_0^2$, i.e., in a similar fashion to (9.104):

$$t_1, s_2 \in \left(\int_{B_{1/4}(p_0)} v_1\, dx, \int_{B_{1/4}(p_0)} w_1\, dx\right), \tag{13.1}$$

and $\widehat{\mathcal{C}}_0^i, i = 1, 2,$ as $\widehat{\mathcal{C}}_0$ in Chapter 9 with $\widehat{s}_i, \widehat{t}_i$ such that $\widehat{v}_1 \in \widehat{\mathcal{C}}_0^1$, $\widehat{w}_1 \notin \widehat{\mathcal{C}}_0^1$, $\widehat{v}_1 \notin \widehat{\mathcal{C}}_0^2$, $\widehat{w}_1 \in \widehat{\mathcal{C}}_0^2$, i.e., similarly to (9.105):

$$\widehat{t}_1, \widehat{s}_2 \in \left(\int_{B_{1/4}(p_0)} \widehat{v}_1\, dx, \int_{B_{1/4}(p_0)} \widehat{w}_1\, dx\right). \tag{13.2}$$

Given $m = (m_1, m_2)$, define $\widehat{Y}_m^i, \widehat{b}_m^i, i = 1, 2$, as in Chapter 9, and

$$\mathcal{M}_{1,m}^i = \left\{u \in \widehat{Y}_m^i \mid J_1(u) = \widehat{b}_m^i\right\},\ i = 1, 2. \tag{13.3}$$

Assume that $m_2 - m_1$ is large enough so that Proposition 9.81 applies. Denote by U_1 the largest element of $\mathcal{M}_{1,m}^1$ and by U_2 the smallest element of $\mathcal{M}_{1,m}^2$. By Corollary 9.95, $U_1 < U_2$, since $f_1 = v_1 < w_1 < f_2$.

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In addition, since there is a gap between $\tau^1_{-1}v_1, \tau^1_{-1}w_1$ and $\tau^1_{-1}\widehat{v}_1, \tau^1_{-1}\widehat{w}_1$, we can take $t_1, \widehat{s}_1$ such that Corollary 9.95 applies and

$$U_2 \le \tau^1_{-1}U_1. \tag{13.4}$$

In order to construct solutions of (PDE) asymptotic in x_2 to a pair of solutions having two transitions in x_1 as in Chapter 9, we assume that $\mathcal{M}_2(v_1, w_1)$ and $\mathcal{M}_2(\widehat{v}_1, \widehat{w}_1)$ satisfy

$$\begin{gathered}\text{there exist adjacent } v_2, w_2 \in \mathcal{M}_2(v_1, w_1) \text{ with } v_2 < w_2,\\ \text{and adjacent } \widehat{v}_2, \widehat{w}_2 \in \mathcal{M}_2(\widehat{v}_1, \widehat{w}_1) \text{ with } \widehat{v}_2 < \widehat{w}_2.\end{gathered} \tag{$*_2$}$$

We now introduce the basic function class for our new solutions. To avoid technical problems in establishing lower bounds for the appropriate analogue of J_2 here, we employ pointwise constraints instead of integral constraints. For this purpose we use constraint functions $g, \widehat{g}$ that are Hölder continuous and satisfy

$$g > v_2 \text{ on } \mathbb{R}^2 \times \mathbb{T}^{n-2}, \tag{g_1}$$

$$g < w_2 \text{ on } B_{\frac{1}{4}}(p_0), \tag{g_2}$$

$$g = \hat{w}_0 \text{ on } \left(\mathbb{R}^2 \times \mathbb{T}^{n-2}\right) \setminus B_{\frac{1}{3}}(p_0), \tag{g_3}$$

and symmetrically,

$$\hat{g} < \hat{w}_2 \text{ on } \mathbb{R}^2 \times \mathbb{T}^{n-2}, \tag{$\hat{g}_1$}$$

$$\hat{g} > \hat{v}_2 \text{ on } B_{\frac{1}{4}}(p_0), \tag{$\hat{g}_2$}$$

$$\hat{g} = v_0 \text{ on } \left(\mathbb{R}^2 \times \mathbb{T}^{n-2}\right) \setminus B_{\frac{1}{3}}(p_0). \tag{$\hat{g}_3$}$$

The class of admissible functions we will use is

$$\mathcal{Y}_m = \left\{u \in W^{1,2}_{\text{loc}}(\mathbb{R}^2 \times \mathbb{T}^{n-2}) \mid u \text{ satisfies (13.5)(i)–(iv)}\right\},$$

where

$$\begin{array}{rl}\text{(i)} & u \le \tau^i_{-1}u,\ i = 1, 2,\\ \text{(ii)} & U_1 \le u \le U_2,\\ \text{(iii)} & v_2 \le \tau^1_{-m_1}u \le g,\\ \text{(iv)} & \hat{g} \le \tau^1_{-m_2}u \le \hat{w}_2.\end{array} \tag{13.5}$$

The renormalized functional J_2 of Chapter 4 was introduced so as to be defined on $\Gamma_2(v_1, w_1)$. Here we seek a heteroclinic in x_2 between members of $\mathcal{M}^1_{1,m}$ and $\mathcal{M}^1_{2,m}$. In general, J_2 will not be defined for such functions, and a variant $\hat{J}_2$

of J_2 is required. To introduce it, observe that using (13.4), the argument in (4.4)–(4.9) with v, w replaced by U_1, U_2, implies that $J_1(\tau^2_{-k}u)$ is well defined for $u \in \mathcal{Y}_m$ and $k \in \mathbb{Z}$. Let

$$\widehat{J}_{2,i}(u) = J_1(\tau^2_{-i}u) - b_i,$$

where $b_i = \widehat{b}^1_m$ for $i < 0$, $b_i = \widehat{b}^2_m$ for $i \geq 0$, and

$$\widehat{J}_{2;p,q}(u) = \sum_{i=p}^{q} \widehat{J}_{2,i}(u). \tag{13.6}$$

Then we have:

Proposition 13.7. *If $m_2 - m_1$ is large enough, $u \in \mathcal{Y}_m$, and $p, q \in \mathbb{Z}$, there exists a $\widehat{K}_2 \geq 0$ that is independent of u, p, q, m, such that*

$$\widehat{J}_{2;p,q}(u) \geq -\widehat{K}_2.$$

Once Proposition 13.7 has been established, we can define

$$\widehat{J}_2(u) = \underline{\lim}_{p\to-\infty, q\to\infty} \widehat{J}_{2;p,q}(u) \tag{13.8}$$

as in Chapter 4 and it follows as in Lemma 2.2 that

$$\widehat{J}_{2;p,q}(u) \leq \widehat{J}_2(u) + 2\widehat{K}_2. \tag{13.9}$$

Some preliminaries are needed to prove Proposition 13.7. The proof of Proposition 4.10, the analogue of Proposition 13.7 in Chapter 4, required Proposition 3.59, which roughly says that the minimizer of J_1 on a class of k-periodic functions is in fact achieved in a class of 1-periodic functions. The corresponding result for $\mathcal{M}_{1,m}$ seems to require lower bounds on $m_2 - m_1$ that are dependent on k, due to the integral constraint used in the definition of $\widehat{Y}_m$. To avoid this difficulty, we defined $\mathcal{Y}_m$ without requiring integral constraints, and now introduce an associated k-periodic function class whose elements will be shown to be 1-periodic for large $m_2 - m_1$, but with no k dependence.

Define

$$P_k = \{u \in W^{1,2}_{\text{loc}}(\mathbb{R}^2 \times \mathbb{T}^{n-2}) \mid \tau^2_{-k}u = u, u \text{ satisfies (13.5)(i)-(ii) with } i = 1 \text{ and (13.10)(i)–(ii) for } i = 0, 1, \ldots, k-1\},$$

where

$$\text{(i)}\ \ u \leq \tau^2_i \tau^1_{m_1} g, \qquad \text{(ii)}\ \ u \leq \tau^2_i \tau^1_{m_2} \hat{w}_2. \tag{13.10}$$

We place additional constraints on $t_1, \widehat{t}_1$:

$$t_1 > \int_{B_{1/4}(p_0)} g\,dx, \quad \widehat{t}_1 > \int_{B_{1/4}(p_0)} \widehat{w}_2\,dx, \tag{13.11}$$

which is possible since $g < w_2 < w_1$ on $B_{1/4}(p_0)$, and $\widehat{w}_2 < \widehat{w}_1$ (see (13.1), (13.2)).

For each $k \in \mathbb{N}$, define $\widehat{Y}^1_{m,k}$ as was $\widehat{Y}^1_m$ but with elements being k-periodic in x_2 and satisfying integral constraints on $\tau_i^2 \tau_{m_j}^1 B_{1/4}(p_0)$, that is,

$$s_1 \le \int_{\tau_i^2 \tau_{m_1}^1 B_{1/4}(p_0)} \min(u, w_0)\,dx \le t_1,$$

$$\widehat{s}_1 \le \int_{\tau_i^2 \tau_{m_2}^1 B_{1/4}(p_0)} \max(u, \widehat{v}_0)\,dx \le \widehat{t}_1, \tag{13.12}$$

for $i = 0, 1, \ldots, k-1$.

Take $m_2 - m_1$ so large that

$$\widehat{v}_0 < U_1 \text{ on } \tau_{m_1}^1 B_{1/4}(p_0). \tag{13.13}$$

This is possible, since if h is the smallest element of $\hat{\mathcal{C}}^1_0$, then $h > \hat{v}_0$, and by Theorem 9.9 and Remark 9.93, $\|U_i - h\|_{L^\infty\left(\tau_{m_2}^1 B_{\frac{1}{4}}(p_0)\right)} \to 0$ as $m_2 - m_1 \to \infty$.

We claim that

$$P_k \subset \widehat{Y}^1_{m,k}, \quad k = 1, 2, \ldots. \tag{13.14}$$

To see this, take $u \in P_k$. By the definition of P_k,

$$U_1 \le u \le \tau_i^2 \tau_{m_1}^1 g < \tau_i^2 \tau_{m_1}^1 w_2 < w_0 \text{ on } \tau_i^2 \tau_{m_1}^1 B_{1/4}(p_0). \tag{13.15}$$

Since $U_1 \in \widehat{Y}^1_m \subset \widehat{Y}^1_{m,k}$ and $u \in P_k$,

$$s_1 \le \int_{\tau_i^2 \tau_{m_1}^1 B_{1/4}(p_0)} \min(u, w_0)\,dx \le \int_{B_{1/4}(p_0)} g\,dx < t_1, \tag{13.16}$$

$i = 0, 1, \ldots, k-1$, by (13.11)–(13.12) and (13.16).

Likewise,

$$U_1 \le u \le \tau_i^2 \tau_{m_2}^1 \hat{w}_2, \tag{13.17}$$

so

$$\hat{s}_1 \le \int_{\tau_i^2 \tau_{m_2}^1 B_{\frac{1}{4}}(p_0)} \max(u, \hat{v}_0) dx \le \int_{B_{\frac{1}{4}}(p_0)} \hat{w}_2\,dx < \hat{t}_1 \tag{13.18}$$

via (13.11)–(13.12) and (13.17). That P_k satisfies the remaining conditions defining $\hat{Y}^1_{m,k}$ follows from (13.5)(i)–(ii).

For $u \in P_k$, let

$$J_2^k(u) = \sum_{i=0}^{k-1} J_1(\tau^2_{-i} u) \tag{13.19}$$

and

$$p_k = \inf_{u \in P_k} J_2^k(u). \tag{13.20}$$

Remark 13.21. It is straightforward to find $u_k^* \in P_k$ minimizing (13.20). The next result shows that if $m_2 - m_1$ is large, we can find such a u_k^* that also lies in P_1. It remains an open question whether this is true for all minimizers $u \in P_k$ of (13.20).

Proposition 13.22. *If $m_2 - m_1$ is large enough, then for any $k \in \mathbb{N}$ there exists $u_k^* \in P_k$ such that $p_k = J_2^k(u_k^*)$. Moreover, $u_k^* \in P_1$ and*

$$p_k = kp_1 = k\hat{b}^1_m. \tag{13.23}$$

Proof. Existence of a u minimizing (13.20) follows from standard arguments. For such a u, define

$$u_1 = \min(u, \tau^2_{-1} u), \quad f_1 = \max(u, \tau^2_{-1} u), \tag{13.24}$$

and iteratively define

$$u_i = \min(u_{i-1}, \tau^2_{-i} u), \quad f_i = \max(u_{i-1}, \tau^2_{-i} u), \quad i = 2, \dots, k-1. \tag{13.25}$$

Note that $u_{k-1} = \min(u, \tau^2_{-1} u, \dots, \tau^2_{-k+1} u)$, so u_{k-1} is 1-periodic in x_2. Also $u_i, f_i \in P_k, i = 1, \dots, k-1$, and $J_2^k(u) + J_2^k(\tau^2_{-1} u) = J_2^k(u_1) + J_2^k(f_1)$, so $J_2^k(u_1) = J_2^k(f_1) = p_k$. Similarly, $J_2^k(u_i) = p_k, i = 2, \dots, k-1$. However $u_k^* \equiv u_{k-1} \in P_1$, so $p_k \geq kp_1$. For $v \in P_1$ such that $p_1 = J_2^1(v)$, we have $v \in P_k$ and $J_2^k(v) = kp_1 \geq p_k$, so $kp_1 = p_k$. Recall that $P_1 \subset \widehat{Y}^1_m$, so $p_1 \geq \widehat{b}^1_m$. By Theorem 9.9 and Remark 9.93, U_1 is L^∞ close to v_1 on $\tau^1_{m_1} B_{\frac{1}{3}}(p_0)$ for large $m_2 - m_1$, and by (g_1), $v_1 < v_2 < g$. Therefore $U_1 \leq \tau^1_{m_1} g$ for such $m_2 - m_1$. But then the definition of g implies

$$U_1 \leq \tau^1_{m_1} g \tag{13.26}$$

on $\mathbb{R}^n$. In addition, Proposition 9.88 implies

$$U_1 < \tau^1_{m_2} \widehat{v}_1 < \tau^1_{m_2} \widehat{w}_2. \tag{13.27}$$

Thus $U_1 \in P_1$ and $p_1 \leq \widehat{b}^1_m$, so $p_1 = \hat{b}^1_m$. We are now ready for the:

Proof of Proposition 13.7. We argue roughly as in the analogous situation in Chapter 4. A difference here is that due to the definition of $\hat{J}_{2,i}$, we have to distinguish the cases $i < 0$ and $i \geq 0$. In the proofs of (4.9) and (4.11), take $u \in \mathcal{Y}_m$ and replace v by U_1, w by $\tau_{-1}^1 U_1$, recalling (13.4) to get first

$$J_1(u) = \hat{b}_m^1 + \frac{1}{2}\|\nabla(u-U_1)\|_{L^2(S_0)}^2 + \int_{S_0}(F(x,u)-F(x,U_1))dx$$
$$+ \int_{S_0\cap\{|x_1|<r\}} \nabla(u-U_1)\cdot\nabla v\,dx$$
$$+ \int_{\partial(S_0\cap\{|x_1|\geq r\})}(u-U_1)\frac{\partial v}{\partial \nu}\,dH^{n-1} - \int_{S_0\cap\{|x_1|\geq r\}}(u-U_1)\Delta v\,dx, \tag{13.28}$$

the latter two integrals bounded independently of r, with zero limits as $r \to \infty$, and then

$$\left| J_{2,i}(u) - \frac{1}{2}\|\nabla(u-U_1)\|_{L^2(S_i)}^2 \right| \leq M_2, \tag{13.29}$$

where M_2 now is a constant independent of i, m. In addition, if in the argument following (4.11), χ is defined for $q < 0$ with U_1 replacing v, then we have $\chi \in P_{q-p+1}$ (recall (13.26), (13.27)), so $\widehat{J}_{2;p,q}(\chi) \geq 0$. Combining this with a similar argument for $p > 0$ and arguing as in the proof of Proposition 4.10 completes the proof of Proposition 13.7.

Next corresponding to Proposition 4.16 we have:

Proposition 13.30. *If $m_2 - m_1$ is large, $u \in \mathcal{Y}_m$, and $\widehat{J}_2(u) < \infty$, then*

$$\widehat{J}_{2,i}(u) \to 0, \quad |i| \to \infty, \tag{13.31}$$

$$\|\tau_{-i}^2 u - U_1\|_{W^{1,2}(S_0)} \to 0,\ i \to -\infty, \tag{13.32}$$

$$\|\tau_{-i}^2 u - U_2\|_{W^{1,2}(S_0)} \to 0,\ i \to \infty, \tag{13.33}$$

$$\widehat{J}_2(u) = \lim_{\substack{p\to-\infty\\ q\to\infty}} \widehat{J}_{2;p,q}(u). \tag{13.34}$$

Proof. Let $u \in \mathcal{Y}_m$ and $\widehat{J}_2(u) < \infty$. From (13.9) and (13.29), we see that $\tau_i^2 u$, $i \in \mathbb{N}$, is bounded in $W_{\mathrm{loc}}^{1,2}(S_0)$, so there is a subsequence that converges weakly in $W_{\mathrm{loc}}^{1,2}(S_0)$, strongly in $L_{\mathrm{loc}}^2(S_0)$, and pointwise almost everywhere. However, $u \leq \tau_{-1}^2 u$, so every subsequence has a convergent subsequence with the same limit. Thus the full sequence converges: $\tau_i^2 u \to h$ as $i \to \infty$ for some $h \in W_{\mathrm{loc}}^{1,2}(S_0)$. Note that $\tau_{i-1}^2 u$, $\tau_i^2 u$ have the same limit, so $\tau_{-1}^2 h = h$, and h is 1-periodic in x_2. Thus it follows from (13.5) that $h \in P_1$, and for $m_2 - m_1$ large, $h \in \widehat{Y}_m^1$ by (13.14).

The analogue of Lemma 4.26 holds in the current setting, the proof following with the same modifications as in the proof of Proposition 13.7. Thus $\liminf_{i\to\infty} J_1(\tau_i^2 u) \geq J_1(h)$, so if $J_1(h) > \widehat{b}_m^1$, then $\widehat{J}_2(u) = \infty$, contrary to assumption. Thus $J_1(h) \leq \widehat{b}_m^1$. However, $h \in \widehat{Y}_m^1$, so $J_1(h) = \widehat{b}_m^1$ and $h \in \mathcal{M}_{1,m}^1$.

However, $U_1 \leq h$ and U_1 is the largest element of $\mathcal{M}^1_{1,m}$, so $h = U_1$. Similarly, $\tau^2_{-i} u \to U_2$ as $i \to \infty$. Estimates similar to (4.24) imply that this convergence is in $L^2(S_0)$ due to (13.4).

The remainder of the proof is similar to that of Proposition 4.16, with Proposition 13.22 playing a crucial role as in the proof of Proposition 13.7.

Define

$$a_m = \inf_{u \in \mathcal{Y}_m} \widehat{J}_2(u). \tag{13.35}$$

We now consider the existence of minimizers of $\widehat{J}_2(u)$ in $\mathcal{Y}_m$. For technical reasons we will make some further assumptions on our basic solutions. Assume

$$\begin{aligned} w_0 \ = \ &\widehat{v}_0, \text{ and the pair } w_1, \widehat{v}_1 \text{ are isolated elements of} \\ &\mathcal{M}_1(v_0, w_0), \mathcal{M}_1(\widehat{v}_0, \widehat{w}_0) \end{aligned} \tag{13.36}$$

respectively, and choose $t_2, \widehat{s}_1$ such that

$$\mathcal{C}^2_0 = \{w_1\}, \quad \widehat{\mathcal{C}}^1_0 = \{\widehat{v}_1\}. \tag{13.37}$$

Now we can state the main result of this chapter, which gives the existence of solutions of (PDE) heteroclinic in x_2 from U_1 to U_2.

Theorem 13.38. *If F satisfies (F_1)–(F_2), $(*)_i$ holds, $i = 0, 1, 2$, (13.36), (13.37) hold, and $m_2 >> m_1$, then*

1^o There is a $\widehat{U}_2 \in \mathcal{Y}_m$ such that $\widehat{J}_2(\widehat{U}_2) = a_m$,
i.e. $\mathcal{M}_{2,m} \equiv \{u \in \mathcal{Y}_m \mid \widehat{J}_2(u) = a_m\} \neq \emptyset$.

2^o Any $U \in \mathcal{M}_{2,m}$ satisfies

(a) U is a solution of (PDE),
(b) $\|U - U_1\|_{C^2(S_i)} \to 0$, $i \to -\infty$,
$\|U - U_2\|_{C^2(S_i)} \to 0$, $i \to \infty$,
i.e., U is heteroclinic in x_2 from U_1 to U_2,
(c) $U_1 < U < \tau^2_{-1} U < U_2$ and $U < \tau^1_{-1} U$, i.e., U is strictly 1-monotone in x_1 and x_2.

3^o $\mathcal{M}_{2,m}$ is an ordered set.

The proof of Theorem 13.38 is rather lengthy. The first step is to show that a_m is finite. The next proposition not only confirms $a_m < \infty$ for $m_2 \gg m_1$, but also gives an asymptotic limit for a_m as $m_2 - m_1 \to \infty$, which is required later in establishing 2^o(a).

Proposition 13.39. *Under the hypotheses of Theorem 13.38, given $\delta > 0$, there is an $M(\delta) > 0$ such that*

$$a_m \leq c_2(v_1, w_1) + c_2(\widehat{v}_1, \widehat{w}_1) + \delta \tag{13.40}$$

for $m_2 - m_1 \geq M(\delta)$.

Proof. We first construct an appropriate element of $\mathcal{Y}_m$. For $R > 0$, let

$$U_3 = \max(U_1, \min(U_2, w_0)) \tag{13.41}$$

and

$$u_1 = \begin{cases} U_1, & x_1 \le m_1 - R, \\ \tau^1_{m_1} v_2, & m_1 - R + 1 \le x_1 \le m_1 + R - 1, \\ U_3, & m_1 + R \le x_1 \le m_2 - R + 1, \\ \tau^1_{m_2} \widehat{w}_2, & m_2 - R + 2 \le x_1 \le m_2 + R - 1, \\ U_2, & m_2 + R \le x_1, \end{cases} \tag{13.42}$$

with the usual interpolation in the remaining intervals. In addition, for $L > 0$ define

$$u_2 = \begin{cases} U_1, & x_2 \le -L - 1, \\ u_1, & -L \le x_2 \le L + 1, \\ U_2, & L + 2 \le x_2, \end{cases}$$

with the usual interpolations. Note that

$$\widehat{J}_2(u_2) = \sum_{i=-L-1}^{L+1} \widehat{J}_{2,i}(u_2). \tag{13.43}$$

Let $u_3 = \max(u_2, \tau^1_{m_1} v_2)$ and $u_4 = \min(u_3, \tau^1_{m_2} \widehat{w}_2)$. We claim that there is a constant $M_1(R, L) > 0$ such that $u_4 \in \mathcal{Y}_m$ for $m_2 - m_1 \ge M_1(R, L)$. We first establish (13.5)(i)–(ii) for u_4. Note that $v_1 < v_2 < w_1$ implies that for any R, L, there is an $\epsilon = \epsilon(R, L)$ such that $v_1 + \epsilon \le v_2 \le w_1 - \epsilon$ on $E_{R,L} := \{|x_1| \le R, |x_2| \le L + 2\}$. Moreover, Theorem 9.9 and Remark 9.93 imply that $\tau^1_{-m_1} U_1$ converges uniformly to v_1 for $|x_1| \le R$ as $m_2 - m_1 \to \infty$. Therefore

$$\tau^1_{-m_1} U_1 < v_2 \text{ on } E_{R,L} \tag{13.44}$$

for $m_2 - m_1$ large. Proposition 9.88 implies

$$\tau^1_{m_1} w_1 \le U_2 \tag{13.45}$$

and

$$U_1 \le \tau^1_{m_2} \widehat{v}_1, \tag{13.46}$$

so by (13.45), (13.41), and (13.46), we have

$$\tau^1_{m_1} v_2 < \tau^1_{m_1} w_1 \le \min(U_2, w_0) \le U_3 \le \max(U_1, w_0) \le \tau^1_{m_2} \widehat{w}_2. \tag{13.47}$$

As for (13.44),

$$\widehat{w}_2 < \tau^1_{-m_2} U_2 \text{ on } E_{R,L} \tag{13.48}$$

for $m_2 - m_1$ large. Combining (13.44), (13.47)–(13.48) implies

$$u_1 \le \tau^1_{-1} u_1 \text{ for } |x_2| \le L + 2. \tag{13.49}$$

In addition, $U_1 \le U_3 \le U_2$ and (13.44)–(13.49) imply

$$u_2 \le \tau^i_{-1} u_2, \quad i = 1, 2, \tag{13.50}$$

from which the claim $u_4 \le \tau^i_{-1} u_4, i = 1, 2$, follows due to the identical monotonicity conditions satisfied by $v_2, \widehat{w}_2$. Moreover, (13.50) for $i = 2$ and the definition of u_2 imply

$$U_1 \le u_2 \le U_2, \tag{13.51}$$

so the inequalities $v_2 \le w_1$ and (13.45) give $U_1 \le u_3 \le U_2$. Then (13.46) and $\widehat{v}_1 \le \widehat{w}_2$ further imply $U_1 \le u_4 \le U_2$. Thus (13.5)(i)-(ii) hold for u_4.

To verify (13.5)(iii)–(iv) and therefore that $u_4 \in \mathcal{Y}_m$, note that by the definitions of u_4 and u_3, $\tau^1_{-m_2} u_4 \le \hat{w}_2$ and $v_2 \le \tau^1_{-m_1} u_3$. Since $u_4 \le u_3$, if $u_4(x) = u_3(x)$, then $v_2(x) \le \tau^1_{-m_1} u_4(x)$, while if $u_4(x) < u_3(x)$, $u_4(x) = \tau^1_{m_2} \hat{w}_2(x) > \tau_j v_2(x)$ for all j. Hence in either event, $v_2 \le \tau^1_{-m_1} u_4$. Thus verifying (13.5)(iii)–(iv) reduces to checking that $\tau^1_{-m_1} u_4 \le g$ and $\widehat{g} \le \tau^1_{-m_2} u_4$. On $B_{\frac{1}{3}}(p_0)$, $\tau^1_{-m_1} u_2 = v_2 = \tau^1_{-m_1} u_4 < g$ via (g_1), while on $(\mathbb{R}^2 \times \mathbb{T}^{n-2}) \backslash B_{\frac{1}{3}}(p_0)$, $g = \widehat{w}_0 \ge \tau^1_{-m_1} u_4$ via (g_3). Thus (13.5)(iii) holds and (13.5)(iv) is verified similarly.

Now we seek to estimate a_m. Recall that $u_3 = \max(u_2, \tau^1_{m_1} v_2)$ and let $f_3 = \min\,(u_2, \tau^1_{m_1} v_2)$. We claim that $\tau^1_{-m_1} f_3 \in \Gamma_2(v_1, w_1)$. To see this, note that Proposition 9.88 implies $\tau^1_{m_1} v_1 \le \min(U_1, \tau^1_{m_1} v_2)$ so by (13.51), $v_1 \le \tau^1_{-m_1} f_3 \le v_2$. Consequently, $\|\tau^1_{-m_1} f_3 - v_1\|_{L^2(S_i)} \to 0$ as $i \to -\infty$. For $i \ge L + 2$, $f_3 = \min(U_2, \tau^1_{m_1} v_2) = \tau^1_{m_1} v_2$ on S_i, since $\tau^1_{m_1} v_2 \le \tau^1_{m_1} w_1 \le U_2$ by Proposition 9.88. Thus $\|\tau^1_{-m_1} f_3 - w_1\|_{L^2(S_i)} \to 0$ as $i \to \infty$. Therefore $\tau^1_{-m_1} f_3 \in \Gamma_2(v_1, w_1)$ and

$$J_2(v_2) \le J_2(f_3). \tag{13.52}$$

Similarly, defining $f_4 = \max(u_3, \tau^1_{m_2} \widehat{w}_2)$, we have $\tau^1_{-m_2} f_4 \in \Gamma_2(\widehat{v}_1, \widehat{w}_1)$ and

$$J_2(\widehat{w}_2) \le J_2(f_4). \tag{13.53}$$

Combining these observations,

$$\begin{aligned} \widehat{J}_{2,i}(u_2) + J_{2,i}(v_2) &= \widehat{J}_{2,i}(u_3) + J_{2,i}(f_3), \\ \widehat{J}_{2,i}(u_3) + J_{2,i}(\widehat{w}_2) &= \widehat{J}_{2,i}(u_4) + J_{2,i}(f_4), \end{aligned} \tag{13.54}$$

and summing over i, we have

$$\widehat{J}_2(u_2) + J_2(v_2) + J_2(\widehat{w}_2) = \widehat{J}_2(u_4) + J_2(f_3) + J_2(f_4). \tag{13.55}$$

Therefore by (13.52), (13.53), and (13.55),

$$\widehat{J}_2(u_4) \le \widehat{J}_2(u_2). \tag{13.56}$$

Since $u_4 \in \mathcal{Y}_m$, (13.43) and (13.56) imply

$$a_m \le \sum_{i=-L-1}^{L+1} \widehat{J}_{2,i}(u_2) \tag{13.57}$$

for $m_2 - m_2 \ge M_1(R, L)$.

Now we will estimate the right-hand side of (13.57) using Proposition 9.107 to aid us. Let J_1^R, S_0^R, etc. be as in that proposition. Likewise, as there, $\kappa_j(\sigma)$ will be used repeatedly to denote functions that go to 0 as $\sigma \to 0$. Let $\sigma > 0$. The parameters R and L will depend on σ.

Note that for $m_1 + R \le x_1 \le m_2 - R + 1$, $-L - 1 \le x_2 \le -L$,

$$u_2 = (x_2 + L + 1)U_3 + (-L - x_2)U_1,$$

so

$$\|\nabla(u_2 - U_1)\|_{L^2(\tau^2_{-L-1}S_0^R)} \le \|\nabla(U_3 - U_1)\|_{L^2(S_0^R)} + \|U_3 - U_1\|_{L^2(S_0^R)} \le \kappa_5(\sigma) \tag{13.58}$$

Combining this with (9.116) using $i = 1$, $u = \tau^2_{L+1}u_2$, and (9.128), (9.129), (9.133), and (9.115), we have $J_1^R(\tau^2_{L+1}u_2) \le \kappa_{13}(\sigma)$. The same is true with $L + 1$ replaced by $-(L + 1)$. In addition, $u_2 = U_3$ for $m_1 + R \le x_1 \le m_2 - R + 1$, $-L \le x_2 \le L + 1$, so due to (9.137) we have

$$J_1^R(\tau_i^2 u_2) \le \kappa_{14}(\sigma), \quad -L - 1 \le i \le L + 1. \tag{13.59}$$

The arguments that gave (9.108)–(9.110) further show that

$$J_{1;-\infty,m_1-R-1}(\tau_i^2 u_2) \le \kappa_{15}(\sigma), \quad J_{1;m_2+R,\infty}(\tau_i^2 u_2) \le \kappa_{15}(\sigma), \tag{13.60}$$

for $-L-1 \le i \le L+1$. In addition, u_2 is close to $v_0, w_0, w_0, \widehat{w}_0$ in $W^{1,2}$ respectively for $x_1 \in [m_1 - R, m_1 - R + 1]$, $[m_1 + R - 1, m_1 + R]$, $[m_2 - R + 1, m_2 - R + 2]$, $[m_2 + R - 1, m_2 + R]$. Arguing as in (9.57)–(9.58), we have

$$b_i \ge c_1(v_0, w_0) + c_1(\widehat{v}_0, \widehat{w}_0). \tag{13.61}$$

Combining (13.57), (13.59), and (13.60) shows that

$$a_m \le \sum_{i=-L-1}^{L+1} \left[J_{1;m_1-R+1,m_1+R-2}(\tau_{-i}^2 u_2) + J_{1;m_2-R+2,m_2+R-2}(\tau_{-i}^2 u_2) - b_i \right] + L\kappa_{16}(\sigma) \tag{13.62}$$

for $R \ge r_3(\sigma)$ and $m_2 - m_1 \ge M_3(R, L)$. Given any $\varepsilon > 0$, we claim that

$$J_{1;m_1-R+1,m_1+R-2}(\tau_{-i}^2 u_2) + J_{1;m_2-R+2,m_2+R-2}(\tau_{-i}^2 u_2) \le b_i + \kappa_{17}(\epsilon) \tag{13.63}$$

for $i = -L-1, L+1$, $R \ge r_4(\varepsilon), L \ge L_0(\varepsilon)$, $m_2 - m_1 \ge M_4(\varepsilon)$. Assuming (13.63) for now, then from the definitions of u_1, u_2 and (13.60)–(13.63), we have

$$a_m \le \sum_{i=-L}^{L} \left[J_{1;-R+1,R-2}(\tau_{-i}^2 v_2) + J_{1;-R+1,R-2}(\tau_{-i}^2 \widehat{w}_2) - c_1(v_0, w_0) - c_1(\widehat{v}_0, \widehat{w}_0) \right] + L\kappa_{16}(\sigma) + 2\kappa_{17}(\epsilon). \tag{13.64}$$

Note that $v_1 \le v_2 \le w_1 \le \tau_{-1}^1 v_1$ and v_1 is L^∞ close to v_0 for $x_1 \le m_1 - R + 2$ for R large, so the same is true for v_2. Estimates like those giving (9.110) then imply that v_2 is close to v_0 in $W^{1,2}$ for say $-p \le x_1 \le -p+1$ and $m_1 - R \le x_1 \le m_1 - R + 1$, where $p \gg R$. Calculations of the type leading to (9.138) then imply

$$J_{1;-\infty,m_1-R}(v_2) \ge -\kappa_{18}(\sigma) \tag{13.65}$$

for $R \ge r_5(\sigma)$. Similarly,

$$J_{1;m_1+R-1,\infty}(v_2) \ge -\kappa_{18}(\sigma). \tag{13.66}$$

Combining (13.65)–(13.66) with similar estimates with $\hat{w}_2$ replacing v_2 and (13.64) imply

$$a_m \le \sum_{i=-L}^{L} (J_{2,i}(v_2) + J_{2,i}(\widehat{w}_2)) + L\kappa_{18}(\sigma) + 2\kappa_{17}(\epsilon). \tag{13.67}$$

Fix $L \ge L_0(\epsilon)$ so that

$$-\epsilon \le \sum_{|i|>L} (J_{2,i}(v_2) + J_{2,i}(\widehat{w}_2)). \tag{13.68}$$

Thus (13.67)–(13.68) imply

$$a_m \le J_2(v_2) + J_2(\widehat{w}_2) + L\kappa_{18}(\sigma) + \kappa_{19}(\epsilon) \tag{13.69}$$

for $m_2 - m_1 \ge M_3(R, L)$, $R \ge r_5(\sigma)$, and $L \ge L_0(\epsilon)$.

Let $\delta > 0$. Choose ϵ such that $\kappa_{17}(\epsilon) \leq \delta/2$. With ϵ and therefore $L_0(\epsilon), r_5(\epsilon)$ so determined, set $L = L_0(\epsilon)$. Next choose σ such that $L\kappa_{18}(\sigma) \leq \delta/2$. Thus for $m_2 - m_1 \geq M_5(R(\epsilon), L_0(\epsilon))$, (13.69) yields (13.40).

All that remains of the proof of Proposition 13.39 is to verify (13.63). Recall that $v_1 < v_2 < w_1 \leq \tau^1_{-1} v_1$, so for $K = K(\epsilon)$ sufficiently large,

$$\int_{-\infty}^{-K} (v_2 - v_1)\, dx_1 \leq \int_{-\infty}^{-K} (\tau^1_{-1} v_1 - v_1)\, dx_1 = \int_{-K}^{-K+1} (v_1 - v_0)\, dx_1 \leq \varepsilon, \quad (13.70)$$

and similarly

$$\int_{K}^{\infty} (v_2 - v_1)\, dx_1 \leq \varepsilon. \quad (13.71)$$

From Theorem 4.40, $\|v_2 - v_1\|_{L^\infty(S_i)} \to 0$ as $i \to -\infty$, so

$$\int_{-K}^{K} (v_2 - v_1) dx_1, \quad \int_{t-1}^{t} (v_2 - v_1) dx_1 \leq \varepsilon \quad (13.72)$$

for $x_2, t \leq -L$, where $L \geq L_0(\epsilon) > 0$. This in conjunction with (13.70)–(13.71) implies

$$\int_{S_{0;-R+1,R-2}} (v_2 - v_1) dx_1 + \int_{\partial S_{0;-R+1,R-2}} (v_2 - v_1) dH^{n-1} \leq \kappa_{19}(\epsilon) \quad (13.73)$$

for $x_2 \in [-L-1, -L]$. Choose $L_0(\varepsilon)$ such that for $L \geq L_0(\varepsilon)$, we further have

$$\left\| \tau^2_L v_2 - v_1 \right\|_{W^{1,2}(S_0)} \leq \varepsilon. \quad (13.74)$$

This is possible due to Proposition 4.16.

As $m_2 - m_1 \to \infty$, $\tau^1_{-m_1} U_1 \to$ some $h \in \mathcal{C}^1_0$ uniformly on $S_{0;-R+1,R-2}$. But v_1 is the largest element of $\mathcal{C}^1_0$, and by Proposition 9.88, $v_1 < \tau^1_{-m_1} U_1$, so $\tau^1_{-m_1} U_1 \to v_1$ uniformly on $S_{0;-R+1,R-2}$. Earlier estimates then imply $\tau^1_{-m_1} U_1 \to v_1$ in $W^{1,2}(S_{0;-R+1,R-2})$ as well. Therefore for $m_2 - m_1 \geq M_4(R, \varepsilon)$,

$$\int_{S_{0;-R+1,R-2}} |\tau^1_{-m_1} U_1 - v_1|\, dx + \int_{\partial S_{0;-R+1,R-2}} |\tau^1_{-m_1} U_1 - v_1|\, dH^{n-1} + \|\tau^1_{-m_1} U_1 - v_1\|_{W^{1,2}(S_{0;-R+1,R-2})} \leq \kappa_{20}(\varepsilon). \quad (13.75)$$

Thus (13.73)–(13.75) imply

$$\int_{S_{0;-R+1,R-2}} |\tau^2_{L+1} v_2 - \tau^1_{-m_1} U_1|\, dx + \int_{\partial S_{0;-R+1,R-2}} |\tau^2_{L+1} v_2 - \tau^1_{-m_1} U_1|\, dH^{n-1} + \|\tau^2_{L+1} v_2 - \tau^1_{-m_1} U_1\|_{W^{1,2}(S_{0;-R+1,R-2})} \leq \kappa_{21}(\varepsilon). \quad (13.76)$$

Note that $\tau^2_{L+1}u_2 - U_1 = x_2(\tau^1_{m_1}\tau^2_{L+1}v_2 - U_1)$ on $S_{0;m_1-R+1,m_1+R-2}$. Therefore

$$\|\nabla(\tau^2_{L+1}u_2 - U_1)\|_{L^2(S_{0;m_1-R+1,m_1+R-2})} \le 2\|\tau^2_{L+1}v_2 - \tau^1_{-m_1}U_1\|_{W^{1,2}(S_{0;-R+1,R-2})}. \tag{13.77}$$

Similarly, the first two terms in (13.76) can be estimated by their analogues with v_2 replaced by u_2. Thus (13.76) holds with v_2 replaced by u_2 and $\kappa_{21}(\varepsilon)$ by $\kappa_{22}(\varepsilon)$. Combining this with (9.116) (taking $u = \tau^2_{L+1}u_2$, $i = 1$, $p = m_1 - R$, $q = m_1 + R + 1$), with (9.138) (replacing σ by ε), and with mild variations on (9.138) (as following (13.64)) for $-\infty < x_1 \le m_1 - R, m_2 + R - 1 \le x_1 < \infty$, yields

$$J_{1;m_1-R+1,m_1+R-2}(\tau^2_{L+1}u_2) \le J_{1;m_1-R+1,m_1+R-2}(U_1) + \kappa_{23}(\varepsilon). \tag{13.78}$$

Similar estimates with v_1 replaced by $\hat{w}_2$ give

$$J_{1;m_2-R+2,m_2+R-2}(\tau^2_{L+1}u_2) \le J_{1;m_2-R+1,m_2+R-2}(U_1) + \kappa_{24}(\varepsilon). \tag{13.79}$$

Combining (13.78)–(13.79) with the estimate of Proposition 9.107 yields (13.63) for $i = L + 1$. Replacing $L + 1$ by $-L - 1$ and arguing as above gives the remaining case of (13.63), and the proof of Proposition 13.39 is complete.

Now we turn to the:

Proof of Theorem 13.38. By Proposition 13.39, $a_m < \infty$. Let $u_k \in \mathcal{Y}_m$ be a minimizing sequence. As in the proof of Proposition 13.30, we see that along a subsequence we have $u_k \to \widehat{U}_2$ weakly in $W^{1,2}(S_i)$, strongly in $L^2(S_i)$, and pointwise almost everywhere for all $i \in \mathbb{Z}$. Note that $\widehat{U}_2 \in \mathcal{Y}_m$ and due to the analogue of Lemma 4.26 here and (13.9), $\widehat{J}_2(\widehat{U}_2) < \infty$. The proof of Proposition 2.50 with alterations as in the proof of Proposition 4.29 and above implies $u_k - \widehat{U}_2 \to 0$ in $W^{1,2}(S_i), i \in \mathbb{Z}$. The analogue of the proof of part (C) in the proof of Theorem 3.2 then implies $\widehat{J}_2(\widehat{U}_2) = a_m$, so $\mathcal{M}_{2,m} \not\equiv 0$.

We now proceed to the proof of 2^o(a) in Theorem 13.38. Assume that 2^o(a) is false for a sequence $m_k = (m_{k,1}, m_{k,2})$ for which $m_{k,2} - m_{k,1}$ is arbitrarily large, i.e., there exist functions $u_k \in \mathcal{M}_{2,m_k}$, $m_{k,2} - m_{k,1} \to \infty$ as $k \to \infty$ with each u_k failing to satisfying (PDE) for at least some point in $\mathbb{R}^n$. We claim that this leads to a contradiction, thus establishing 2^o(a) for sufficiently large $m_{k,2} - m_{k,1}$, as required.

Proposition 13.39 shows that as $k \to \infty$,

$$\widehat{J}_2(u_k) = a_{m_k} \to c_2(v_1, w_1) + c_2(\widehat{v}_1, \widehat{w}_1). \tag{13.80}$$

Thus from (13.9), for all $k \in \mathbb{N}$ and $p, q \in \mathbb{Z}$,

$$\widehat{J}_{2;p,q}(u_k) \le M_1, \qquad \text{for some } M_1 > 0. \tag{13.81}$$

Take $p = q = i$ in (13.81) and apply the definition of $\hat{J}_{2,i}$ and (2.23), recalling that $\widehat{b}^i_{m_k} \to c_1(v_0, w_0) + c_1(\widehat{v}_0, \widehat{w}_0)$ as $k \to \infty$, to get

$$J_{1;p,q}(\tau^2_{-i}u_k) \le M_2 \tag{13.82}$$

with M_2 independent of p, q, k, and i. Thus $\tau^1_{-m_{k,1}} u_k$ is bounded in $W^{1,2}_{\text{loc}}$, so there is a $\bar{u} \in W^{1,2}_{\text{loc}}$ such that as $k \to \infty$, on a subsequence we have $\tau^1_{-m_{k,1}} u_k \to \bar{u}_1$ weakly in $W^{1,2}_{\text{loc}}$, strongly in L^2_{loc}, and pointwise almost everywhere for some function $\bar{u}_1$. Thus from (13.5),

$$\bar{u}_1 \le \tau^i_{-1}\bar{u}_1, \quad i = 1, 2; \quad v_1 \le \bar{u}_1 \le w_1, \quad v_2 \le \bar{u}_1 \le g, \tag{13.83}$$

the second inequality implied by $\tau^1_{-m_{k,1}} U_{1,k} \to v_1$, $\tau^1_{-m_{k,1}} U_{2,k} \to w_1$ as $k \to \infty$. Here $U_{1,k}$, $U_{2,k}$ are the U_1, U_2 associated with problem k.

The lower semicontinuity of J_1 and (13.81) with $p = q = i$ imply $J_1(\tau^2_{-i}\bar{u}_1) \le M_3 < \infty$, so (4.8) implies $\|\nabla(\bar{u}_1 - v_1)\|_{L^2(S_i)} < \infty$ for all i. As in the proof of Proposition 13.30, the monotonicity conditions $\tau^2_{-i} u_1 \ge u_1$ and the fact that v_1, w_1 is a gap pair with $v_1 \le \bar{u}_1 \le w_1$ imply that there are functions $\psi^+ \in \{v_1, w_1\}$ such that

$$\tau^2_{-i}\bar{u}_1 \to \psi^{\pm} \text{ in } L^2(S_0) \text{ as } i \to \pm\infty. \tag{13.84}$$

By (13.83), $v_1 < v_2 \le \bar{u}_1 \le g$, so $\psi^+ = w_1$. By (g_2), $\bar{u}_1 \not\equiv w_1$, so $\psi^- = v_1$. Thus

$$\bar{u}_1 \in \Gamma_2(v_1, w_1). \tag{13.85}$$

From (13.8), (13.81) we have $\widehat{J}_2(\bar{u}_1) \le M_1$, so Proposition 4.16 applies to $\bar{u}_1$, and the limits in (13.84) are in $W^{1,2}(S_0)$ as well.

In the same manner we can assume that as $k \to \infty$ along our subsequence,

$$\tau^1_{-m_{k,2}} u_k \to \bar{u}_2 \in \Gamma_2(\widehat{v}_1, \widehat{w}_1) \tag{13.86}$$

with $\widehat{J}_2(\bar{u}_2) < \infty$ and $\|\nabla(\bar{u}_2 - \widehat{v}_1)\|_{L^2(S_i)} < \infty$ for all i.

In order to study the convergence of u_k more carefully, it is necessary to establish that u_k satisfies (PDE) in certain regions. To do so, we use a variant of an argument from the proof of Theorem 9.6. Given $p \in T_0$, $r > 0$ such that $B_{2r}(p) \subseteq T_0$, let $B_{i,j}(r) = \tau^1_i \tau^2_j B_r(p)$, $B = \bigcup_{i,j \in \mathbb{Z}} B_{i,j}(r)$, and

$$\widetilde{u_k} = \begin{cases} u_k, & x \in \mathbb{R}^n \setminus B, \\ u_{i,j,k}, & x \in B_{i,j}(r), \end{cases}$$

where $u_{i,j,k}$ are defined as minimizers of the following variational problem. For $m_{k,1} < i < m_{k,2}$, $j \in \mathbb{Z}$, let $u_{i,j,k} \in W^{1,2}(B_{i,j}(2r))$ be the largest minimizer of

$$I_{i,j}(u) = \int_{B_{i,j(2r)}} L(u)\, dx \tag{13.87}$$

over all $u \in \mathfrak{F}_{i,j}$, where

$$\mathfrak{F}_{i,j} = \left\{ u \in W^{1,2}(B_{i,j}(2r)) \,|\, u = u_k \text{ on } B_{i,j}(2r)/B_{i,j}(r) \right\}.$$

Use the same definition of $u_{i,j,k}$ for $i \geq m_{k,2}$, $j < 0$ and for $i \leq m_{k,1}$, $j > 0$. For $i \leq m_{k,1}$, $j \leq 0$, we impose the additional restriction to the definition of $\mathcal{F}_{i,j}$ that

$$u \leq \tau_i^1 \tau_j^2 g =: g_{i,j}, \tag{13.88}$$

while for $i \geq m_{k,2}$, $j \geq 0$, we further require

$$u \geq \tau_i^1 \tau_j^2 \widehat{g} =: \widehat{g}_{i,j}. \tag{13.89}$$

The motivation for (13.88), (13.89) is the fact that $u \in \mathcal{Y}_m$ implies (13.88)–(13.89) due to the $g, \widehat{g}$ constraints in the definition of $\mathcal{Y}_m$ and the condition $u \leq \tau_{-1}^i u$, $i = 1, 2$. As, e.g., in Proposition 2.2 for each i, j, the set of minimizers of (13.87) in $\mathcal{F}_{i,j}$ is ordered. Therefore there is a unique largest one $u_{i,j,k}$, so $\tilde{u}_k$ is well defined.

Proposition 13.90. $\tilde{u}_k \in \mathcal{Y}_{m_k}$.

Proof. By construction, $\tilde{u}_k \in W^{1,2}_{\text{loc}}(\mathbb{R}^2 \times \mathbb{T}^{n-2})$. We must verify that $\tilde{u}_k$ satisfies (13.5)(i)–(iv). This need only be done for $x \in B$, since $u_k \in \mathcal{Y}_{m_k}$ and $\tilde{u}_k = u_k$ for $x \in \mathbb{R}^n \backslash B$. We begin with (13.5)(ii). Note that by (13.5)(ii) for u_k, $U_{1,k} \leq u_k$ on $\mathbb{R}^n \backslash B$. Fix $(i, j) \in \mathbb{Z}^2$ and set

$$\varphi = \begin{cases} \tilde{u}_k, & x \in \mathbb{R}^n \backslash B_{i,j}(r), \\ \max(U_{1,k}, \tilde{u}_k), & x \in B_{i,j}(r), \end{cases} \tag{13.91}$$

and

$$\psi = \begin{cases} U_{1,k}, & x \in \mathbb{R}^n \backslash B_{i,j}(r), \\ \min(U_{1,k}, \tilde{u}_k), & x \in B_{i,j}(r). \end{cases}$$

Since $U_{1,k} \leq u_k = \tilde{u_k}$ on $\mathbb{R}^n \backslash B$, an equivalent but simpler definition is $\varphi = \max(U_{1,k}, \tilde{u}_k)$. Similarly $\psi = \min(U_{1,k}, \tilde{u}_k)$. By the local minimality property of $U_{1,k}$ (see, e.g., Remark 9.55),

$$I_{ij}(\psi) \geq I_{ij}(U_{1,k}). \tag{13.92}$$

We claim that $\varphi \in \mathcal{F}_{i,j}$. Assuming this for now,

$$I_{ij}(\varphi) \geq I_{ij}(u_{ijk}). \tag{13.93}$$

But

$$I_{ij}(\varphi) + I_{ij}(\psi) = I_{ij}(U_{1,k}) + I_{ij}(u_{ijk}), \tag{13.94}$$

so by (13.92)–(13.94) $I_{ij}(\psi) = I_{ij}(U_{1,k})$ and $I_{ij}(\varphi) = I_{ij}(u_{ijk})$. Therefore φ is a minimizer of I_{ij} on $\mathcal{F}_{ij}$, and by (13.91), $\varphi \geq u_{ijk} = \tilde{u}_k$ on $B_{ij}(r)$. Since u_{ijk} is the largest of the minimizers of I_{ij} on $\mathcal{F}_{ij}$, $\varphi = u_{ijk}$ on $B_{ij}(r)$. This being true for all $(i, j) \in \mathbb{Z}^2$, $\tilde{u}_k \geq U_{1,k}$ on B. Similarly, $\tilde{u}_k \leq U_{2,k}$ on B, so (13.5)(ii) is valid once we show that $\varphi \in \mathfrak{F}_{ij}$. This is immediate for $m_{k,1} < i < m_{k,2}$ and all $j \in \mathbb{Z}$, for $i \geq m_{k,2}$ and $j < 0$, and for $i \leq m_{k,1}$ and $j > 0$, since there are no further constraints on φ for these cases. The two remaining cases are (α) $i \leq m_{k,1}$, and $j \leq 0$ and (β) $i \geq m_{k,2}$ and $j \geq 0$. For (α), minimizers of $\mathfrak{F}_{ij}$ must satisfy the further condition $u \leq g_{ij}$. Therefore $u_{ijk} \leq g_{ij}$. But by (13.5)(i) again, $\varphi = \max(U_{1,k}, u_{ijk}) \leq \max(u_k, g_{ij})$ on $B_{ij}(r)$, and by (13.5)(i) and (iii), $u_k \leq \tau^1_{i-m_{k,1}} \tau^2_j u_k \leq g_{ij}$. Thus $\varphi \in \mathfrak{F}_{ij}$. A similar argument holds for (β), so (13.5)(ii) has been verified for $\tilde{u}_k$.

Next we verify (13.5)(iii). Since v_2 has a local minimality property, $v_2 \leq \tau^1_{-m_{1,k}} \tilde{u}_k$ follows as did (13.5)(ii). Also $\tau^1_{-m_1,k} \tilde{u}_k \leq g$ on T_0 is built into the definition of $\mathfrak{F}_{ij}$. This holds on the rest of $\mathbb{R}^n$ due to the minimality property of $\hat{w}_0$ and (g_3). The condition (13.5)(iv) holds for similar reasons.

It remains only to check (13.5)(i):

$$\tilde{u}_k \leq \tau^i_{-1} \tilde{u}_k, \quad i = 1, 2. \tag{13.95}$$

This is a consequence of the following lemma:

Lemma 13.96. *Assume $\bar{g}$, f_1, $f_2 \in W^{1,2}(B_{i,j}(2r))$ with $f_1 \leq f_2$ on $B_{i,j}(2r)/B_{i,j}(r)$ and that u_ℓ is the largest (smallest) minimizer of $I_{i,j}(u)$ over*

$$A_\ell = \{u \in W^{1,2}(B_{i,j}(2r)) \mid u \geq \bar{g} \text{ in } B_{i,j}(2r) \text{ and } u = f_\ell \text{ on } B_{i,j}(2r)/B_{i,j}(r)\},$$

$\ell = 1, 2$. *Then $u_1 \leq u_2$.*

Proof. Suppose $u_\ell \in A_\ell$ is the largest minimizer of I_{ij}, $\ell = 1, 2$. Let $v_1 = \min(u_1, u_2)$, $v_2 = \max(u_1, u_2)$, so $v_1 \leq v_2$, $v_\ell \in A_\ell$, $\ell = 1, 2$, and

$$J_{i,j}(v_1) + J_{i,j}(v_2) = J_{i,j}(u_1) + J_{i,j}(u_2).$$

Thus $J_{i,j}(v_\ell) = J_{i,j}(u_\ell)$, $\ell = 1, 2$, and $v_2 \geq u_2$. Consequently $v_2 = u_2$, and so $v_1 = u_1$. Thus $u_1 \leq u_2$. In the case that $u_\ell, \ell = 1, 2$, are the smallest minimizers, then again $v_1 \leq u_1$, so $v_1 = u_1$. Thus $v_2 = u_2$, and again $u_1 \leq u_2$.

Slight variations on the proof of Lemma 13.96 then give the following three lemmas:

Lemma 13.97. *If the condition $u \geq \bar{g}$ is dropped from A_1, $u_2 \geq u_1$.*

Lemma 13.98. *Lemma 13.96 holds with the condition $u \geq \bar{g}$ in A_ℓ replaced by $u \leq \bar{g}$, $\ell = 1, 2$. In addition, Lemma 13.97 holds with the condition $u \geq \bar{g}$ dropped from the definition of A_2, and the condition $u \leq \bar{g}$ added to A_1.*

Lemma 13.99. *Lemma 13.96 holds with the condition $u \geq \bar{g}$ dropped from the variational problems defining $u_\ell, \ell = 1, 2$.*

Now to complete the proof of Proposition 13.90, note from the definition of $\widetilde{u}_k$ that (13.95) holds on $\mathbb{R}^n \backslash B$. Set $f_1 = u_k$, $f_2 = \tau^1_{-1} u_k$, $u_1 = u_{i,j,k}$, $u_2 = \tau^1_{-1} u_{i+1,j,k}$,

$\bar{g} = g_{i,j}$. For i, j such that $m_{k,1} < i < m_{k,2} - 1$, $j \in \mathbb{Z}$, apply Lemma 13.99 to get $\widetilde{u}_k \le \tau_{-1}^1 \widetilde{u}_k$ on $B_{i,j}(r)$. For $i = m_{k,2} - 1$, $j \ge 0$, apply Lemma 13.97. Applying Lemmas 13.96–13.99 appropriately in the various remaining cases gives (13.95) for $i = 1$. Replacing τ_{-1}^1 by τ_{-1}^2 in the above establishes (13.95) for $i = 2$.

Now that $\tilde{u}_k \in \mathcal{Y}_{m_k}$ has been established, we return to the proof of Theorem 13.38. Note that

$$\widehat{J}_2(\widetilde{u}_k) \le \widehat{J}_2(u_k), \tag{13.100}$$

since $u_k \in \mathcal{F}_{ij}$ for all i, j, k and therefore $J_{ij}(u_{i,j,k}) \le J_{ij}(u_k)$. By (13.100), $\widetilde{u}_k \in \mathcal{M}_{2,m_k}$ and

$$\widehat{J}_2(\widetilde{u}_k) = \widehat{J}_2(u_k). \tag{13.101}$$

Moreover, u_k is a minimizer of $I_{i,j}$ over $\mathcal{F}_{i,j}$ for each $i, j \in Z$. Consider those i, j for which $\mathcal{F}_{ij}$ does not include a $u \le g_{i,j}$ or $u \ge \widehat{g}_{i,j}$ constraint. Standard arguments then imply that u_k is a solution of (PDE) in $B_{i,j}(r)$. We will show that u_k is the unique solution of I_{ij} in $\mathcal{F}_{ij}$ for this case.

To see this, for $0 < \theta < 2$ repeat the above construction with $\mathcal{F}_{i,j}$ replaced by $\mathcal{F}_{i,j,\theta}$, the analogue of $\mathcal{F}_{i,j}$, with $B_{i,j}(r)$ replaced by $B_{i,j}(\theta r)$ but $B_{i,j}(2r)$ remaining the same. Fixing i and j, let $\widetilde{u}_{k,\theta}$ be the largest minimizer of $I_{i,j}$ over $\mathcal{F}_{i,j,\theta}$. For $0 < \theta_1 < \theta_2 < 2$, by (13.101) we have

$$I_{i,j}(\widetilde{u}_{k,\theta_2}) = I_{i,j}(u_k) = I_{i,j}(\widetilde{u}_{k,\theta_1}).$$

But $\tilde{u}_{k,\theta_1}, u_k \in \mathcal{F}_{i,j,\theta_2}$. Thus $\tilde{u}_{k,\theta_1}, u_k$ are solutions of (PDE) in $B_{ij}(\theta_2 r)$ with $\tilde{u}_{k,\theta_1} = u_k$ in $B_{i,j}(\theta_2 r)\backslash B_{i,j}(\theta_1 r)$ and $\tilde{u}_{k,\theta_1} \ge u_k$. Thus by the maximum principle, $\tilde{u}_{k,\theta_1} = u_k$. Let $\hat{u}_{k,\theta_1}$ be the smallest minimizer of $I_{i,j}$ over $\mathcal{F}_{i,j,\theta}$. Since $I_{i,j}(\tilde{u}_{k,\theta_1}) = I_{j,k}(\hat{u}_{k,\theta_1})$, the above argument implies $u_k = \hat{u}_{k,\theta_1}$. Thus u_k is the unique minimizer of $I_{i,j}$ over $\mathcal{F}_{i,j,\theta}, 0 < \theta < 2$, as claimed above.

For the next step in proving that u_k is a solution of (PDE), consider balls B_{ij}, where $\mathcal{F}_{ij}$ has a constraint. Note that by ($\hat{g}_1$)–($\hat{g}_3$), $\widehat{g} \le \widehat{w}_0 - \varepsilon$ for some $\varepsilon > 0$. In addition, $\tau_{-m_{k,2}-\ell} U_{1,k} \to \widehat{w}_0$ on T_0 as $\ell \to \infty$ independently of m_k via Theorem 9.6. Hence there is an $l_0 \in \mathbb{N}$ such that $\widehat{g} \le \tau^1_{-m_{k,2}-\ell} U_{1,k} - \varepsilon/2$ in T_0 for $\ell \ge \ell_0$. Thus

$$\widehat{g}_{i,j} + \varepsilon/2 \le U_{1,k} \tag{13.102}$$

on $T_{i,j} := \tau_i^1 \tau_j^2 T_0$ for $i \ge m_{k,2} + \ell_0$. Let $\bar{u}_{i,j,k}$ be the largest minimizer of $I_{i,j}$ over $\mathcal{F}_{i,j}$ but with the $\widehat{g}_{i,j}$ constraint dropped. Since $u_k \ge U_{1,k}$, $\bar{u}_{ijk} \ge U_{1,k}$ on $\mathbb{R}^n \backslash B_{ij}(r)$, by a familiar argument (see, e.g., (A) of the proof of Theorem 3.2),

$$\bar{u}_{i,j,k} \ge U_1 > \widehat{g}_{i,j}.$$

Hence in fact $\bar{u}_{i,j,k}$ is the largest minimizer of the variational problem with the $\hat{g}_{ij}$ constraint and $\bar{u}_{ijk} = u_{ijk}$. Consequently, $u_{i,j,k}$ is a minimum of the unconstrained variational problem, as is u_k, so by the argument of the previous paragraph, $u_{i,j,k} = u_k$ on $B_{i,j}(r)$, and u_k satisfies (PDE) on $B_{i,j}(r)$ for $i \ge m_{k,2} + \ell_0$. We can assume that the same is true for $i \le m_{k,1} - \ell_0$. This leaves the cases $m_{k,1} - \ell_0 < i \le m_{k,2}$ and $j \le 0$, and $m_{k,1} \le i < m_{k,2} + \ell_0$ and $j \ge 0$ still to be checked.

The restriction $B_{2r}(p) \subset T_0$ can be removed, for example by considering the variational problem in strips of width wider than one. Thus u_k satisfies (PDE) for $x_1 \leq m_{k,1} - \ell_0$, for $x_1 \geq m_{k,2} + \ell_0$, for $m_{k,1} + 1 \leq x_1 \leq m_{k,2}$ and for $x_1 \leq m_{k,1} + 1$, $x_2 \geq 1$, and $x_1 \geq m_{k,2}$, $x_2 \leq 0$. To treat the remaining regions, as before take p_0 to be the center of T_0.

Lemma 13.103. *Let $f \in W^{1,2}(T_0)$, and let $\bar{g}$ be Hölder continuous on T_0. Suppose $\bar{u}$ is a minimizer of*

$$\int_{B_{5/12}(p_0)} L(u)\, dx \quad \text{over } u \in \mathcal{F},$$

where

$$\mathcal{F} = \{u \in W^{1,2}(T_0) \,|\, u = f \text{ on } T_0 \backslash B_{5/12}(p_0), u \geq \bar{g} \text{ on } B_{5/12}(p_0)\}.$$

Then $\bar{u}$ is Hölder continuous on $\overline{B_{1/3}(p_0)}$ with the Hölder exponent and constant dependent only on F, $\bar{g}$.

Proof. This follows from Theorem 3.7 of Michael and Ziemer [32], since the estimates there depend only on structure conditions and the distance between the domain boundary and the set on which the Hölder continuity estimate is required.

Remark 13.104. Given that $\bar{u}, \bar{g}$ are Hölder continuous, they satisfy a common modulus of continuity estimate $|u(x) - u(y)| \leq \varepsilon$ for $|x - y| \leq \delta(\varepsilon)$, $\delta(\varepsilon) = c\varepsilon^{\alpha}$, $\alpha > 1$.

Lemma 13.105. *If $\bar{u}$ and $\bar{g}$ are as in Lemma 13.103, $\delta(\varepsilon)$ as in Remark 13.104, and $v \in L^1(B_{1/3}(p_0))$ with $v \geq \bar{g} + 2\varepsilon$ on $B_{1/3}(p_0)$ and*

$$\|v - \bar{u}\|_{L^1(B_{1/3}(p_0))} \leq \frac{\varepsilon |B_{\delta(\varepsilon)}|}{5}, \tag{13.106}$$

then $\min(\bar{u} - \bar{g}) > 0$ *on $\overline{B_{1/3}(p_0)}$.*

Proof. If not, $\bar{u}(q_0) = \bar{g}(q_0)$ for some $q_0 \in \overline{B_{1/3}(p_0)}$. We can assume that $\bar{u}, \bar{g}$ and $\bar{u} - \bar{g}$ have same modulus of continuity in $B_{\frac{1}{3}}(p_0)$. From Lemma 13.103 we have $\bar{u} - \bar{g} \leq \varepsilon$ and $v - \bar{u} = (v - \bar{g}) - (\bar{u} - \bar{g}) \geq 2\varepsilon - \varepsilon = \varepsilon$ in $B_{\delta(\varepsilon)}(q_0) \cap \overline{B_{1/3}(p_0)}$ by the hypothesis satisfied by v. Thus

$$\|v - \bar{u}\|_{L^1(B_{1/3}(p_0))} \geq \frac{\varepsilon |B_{\delta(\varepsilon)}(q_0)|}{4}, \tag{13.107}$$

since at least one quarter of $B_{\delta(\varepsilon)}$ lies in $B_{1/3}(p_0)$. However, (13.107) contradicts (13.106). Thus $\bar{u} \neq \bar{g}$ on $\overline{B_{1/3}(p_0)}$. Since $\bar{u}$, $\bar{g}$ are continuous on $\overline{B_{1/3}(p_0)}$, $\bar{u} > \bar{g}$, so the proof is complete.

Returning to the proof of Theorem 13.38, recall that from (13.86), $\bar{u}_2 \in \Gamma_2(\widehat{v}_1, \widehat{w}_1)$. Therefore

$$\|\tau^2_{-j}\bar{u}_2 - \widehat{w}_1\|_{W^{1,2}(S_0)} \to 0 \quad \text{as } j \to \infty. \tag{13.108}$$

Since $\hat{w}_1 > \hat{w}_2 \geq \hat{g}$ via ($\hat{g}_1$),

$$\varepsilon = \frac{1}{2}\min_{T_0}(\widehat{w}_1 - \widehat{g}) > 0. \tag{13.109}$$

Define σ by

$$\sigma = \frac{\varepsilon|B_{\delta(\varepsilon)}|}{5}. \tag{13.110}$$

By (13.108)–(13.110), we can pick $j_1 > 0$ such that

$$\|\tau^2_{-j_1}\bar{u}_2 - \widehat{w}_1\|_{W^{1,2}(S_0)} \leq \frac{\sigma}{2}. \tag{13.111}$$

Using L^2_{loc} convergence in (13.86), choose k such that

$$\|\tau^1_{-m_{k,2}}\tau^2_{-j_1}u_k - \tau^2_{-j_1}\bar{u}_2\|_{L^2(S_0\cap\{0\leq x_1\leq \ell_0\})} \leq \frac{\sigma}{2}. \tag{13.112}$$

Thus (13.111) implies

$$\|\tau^1_{-m_{k,2}}\tau^2_{-j_1}u_k - \widehat{w}_1\|_{L^2(S_0\cap\{0\leq x_1\leq \ell_0\})} \leq \sigma. \tag{13.113}$$

Repeat the construction defining $\tilde{u}_k$ with $T_{i,j} := \tau^1_i\tau^2_j T_0$ replacing $B_{i,j}(2r)$ and $B_{i,j}(5/12)$ replacing $B_{i,j}(r)$. This shows that for each $i, j \in \mathbb{Z}$, u_k is a minimizer of a variational problem for a class of functions on $B_{i,j}(5/12)$. In particular, for $j \geq 0$, $m_{k,2} \leq i \leq m_{k,2} + \ell_0$, u_k minimizes $\int_{B_{i,j}(5/12)} L(u)\,dx$ over $u \in \mathcal{F}^*_{i,j}$, where

$$\mathcal{F}^*_{i,j} = \{u \in W^{1,2}(T_{i,j}) \mid u = u_k \text{ on } T_{i,j}\backslash B_{i,j}(5/12), u \geq \widehat{g}_{i,j} \text{ on } B_{i,j}(5/12)\}.$$

Thus $u^*_{i,j,k} := \tau^1_{-i}\tau^2_{-j}u_k$ minimizes $\int_{B_{5/12}(p_0)} L(u)\,dx$ over

$$\mathcal{F}^* = \{u \in W^{1,2}(T_0) \mid u = u^*_{i,j,k} \text{ on } T_0\backslash B_{5/12}(p_0), u \geq \widehat{g} \text{ on } B_{5/12}(p_0)\}.$$

Observe that Lemma 13.105 applies here with $\bar{u} = u^*_{i,j,k}$, $j < j_1$, $f = u^*_{i,j,k}$, and $\bar{g} = \widehat{g}$. Set $v = \tau^1_{m_{k,2}-i}\widehat{w}_1$ so $v \geq \widehat{w}_1 \geq \widehat{g} + 2\varepsilon$ on T_0. Then by (13.110)–(13.109), and (13.112), $\|u^*_{i,j_1,k} - \tau^1_{m_{k,2}-i}\widehat{w}_1\|_{L^2(T_0)} \leq \sigma$ for $m_{k,2} \leq i \leq m_{k,2} + \ell_0$. Hence by Lemma 13.105, $u^*_{i,j_1,k} \geq \widehat{g} + \varepsilon_0$ on $\overline{B_{1/3}(p_0)}$ for some $\varepsilon_0 > 0$. Note that $u^*_{i,j,k} := \tau^1_{-i}\tau^2_{-j}u_k \geq \tau^1_{-i}U_1 \geq \tau^1_{-m_{k,2}}U_1$, the latter inequality due to

$\tau^1_{-1}U_1 \geq U_1$ and $i \geq m_{k,2}$. But Theorem 9.9 implies $\tau^1_{-m_{k,2}}U_1 \geq v_0 + \varepsilon_1$ on T_0 for some $\varepsilon_1 > 0$ and $m_{k,2} - m_{k,1}$ large, since $\widehat{v}_1 > \widehat{v}_0 > v_0$ on T_0. In addition, $\widehat{g} = v_0$ on $T_0 \backslash B_{1/3}(p_0)$. Thus $u^*_{i,j_1,k} \geq \widehat{g} + \varepsilon_1$ on $T_0 \backslash B_{1/3}(p_0)$ and $u^*_{i,j_1,k} \geq \widehat{g} + \varepsilon_3$ on T_0, where $\varepsilon_3 = \min(\varepsilon_0, \varepsilon_1) > 0$. This implies $u_k \geq \widehat{g}_{i,j_1} + \varepsilon_3$ on T_{i,j_1}. Since $\tau^2_{-1}u_k \geq u_k$, it follows that $u_k \geq \widehat{g}_{i,j} + \varepsilon_3$ for $j \geq j_1, m_{k,2} \leq i \leq m_{k,2} + \ell_0$. Consequently, standard arguments now imply that u_k satisfies (PDE) on $T_{i,j}$, $j \geq j_1, m_{k,1} - \ell_0 \leq i \leq m_{k,1}$. Combined with earlier results, this implies that u_k satisfies (PDE) for $x_2 \geq j_1$. By an analogue of Lemma 13.105 for g and related arguments, we can assume that u_k satisfies (PDE) for $x_2 \leq -j_1$. Letting $k \to \infty$ and taking limits in the weak formulation of (PDE) then implies that $\bar{u}_1$ satisfies (PDE) for $|x_2| \geq j_1$ and for $x_1 \leq -\ell_0, x_1 \geq 1$, while $\bar{u}_2$ satisfies (PDE) for $|x_2| \geq j_1$, and for $x_1 \leq 0, x_1 \geq \ell_0$.

Next we will show that for large k, u_k satisfies (PDE) for all $x \in \mathbb{R}^n$. We claim that

$$c_2(v_1, w_1) + c_2(\hat{v}_1, \hat{w}_1) \geq J_2(\bar{u}_1) + J_2(\bar{u}_2). \tag{13.114}$$

But $\bar{u}_1 \in \Gamma_2(v_1, w_1)$ and $\bar{u}_2 \in \Gamma_2(\hat{v}_1, \hat{w}_1)$. Hence (13.114) shows that $\bar{u}_1 \in \mathcal{M}_2(v_1, w_1)$ and $\bar{u}_2 \in \mathcal{M}_2(\hat{v}_1, \hat{w}_1)$. By (13.83), $v_2 \leq \bar{u}_1 \leq g$, so ($g_2$) implies $\bar{u}_1 < w_2$. Therefore v_2, w_2 being a gap pair in $\mathcal{M}_2(v_1, w_1)$, it must be the case that $\bar{u}_1 = v_2$. Similarly $\bar{u}_2 = \hat{w}_2$. Thus by ($\hat{g}_1$), $\bar{u}_2 > \hat{g}$. Note that Lemma 13.103 applies to $\tau^1_{-m_k,2}u_k \equiv \bar{u}$ with $f = \bar{u}$ and $\bar{g} = \hat{g}$. Since $\tau^1_{-m_k,2}u_k \to \bar{u}_2$ in $L^2(T_0)$ along a subsequence of $k \to \infty$, taking $v = \bar{u}_2$ in Lemma 13.105 shows that $\tau^1_{-m_{k,2}}u_k > \hat{g}$ on T_0 for large k. With this strict inequality, standard arguments then imply that $\tau^1_{-m_{k,2}}u_k$ satisfies (PDE) on T_0. Note that$\tau^\ell_{-1}u_k \geq u_k, \ell = 1, 2$ implies that $u_k > \widehat{g}_{i,j}$ on $T_{i,j}$, $i \geq m_{k,2}$, $j \geq 0$, so the same arguments imply u_k satisfies (PDE) for $x_2 \geq 0$. Similarly, $u_k < g_{i,j}$ on $T_{i,j}$, $i \leq m_{k,1}$, $j \leq 0$, so u_k satisfies (PDE) in $\mathbb{R}^n$. This is contrary to our original assumption so 2^o(a) is established once we verify the claim (13.114). To do so requires three steps: (a) Given any $\varepsilon > 0$, there are a $j_2(\varepsilon) \geq j_1$ and $k_1 = k_1(\varepsilon, j)$ such that

$$\hat{J}_2(u_k) \geq \sum_{|i| \leq j} J_{2,i}(u_k) - \varepsilon \tag{13.115}$$

for $j \geq j_2(\varepsilon)$ and $k \geq k_1(\varepsilon, j)$; (b) With j and k so chosen, for any $\delta > 0$, there are an $R_0 = R_0(\delta) > l_0$ and $k_2(\delta)$ such that if $R \geq R_0(\delta)$ and $k \geq \max(k_1(\varepsilon, j), k_2(\delta))$,

$$\begin{aligned}\hat{J}_2(u_k) \geq \sum_{|i| \leq j} \Big[&\Big(J_{1;-R,R}(\tau^2_{-i}\tau^1_{-m_{k,1}}u_k) - c_1(v_0, w_0) \Big) \\ &+ \Big(J_{1;-R,R}(\tau^2_{-i}\tau^1_{-m_{k,2}}u_k) - c_1(\hat{v}_0, \hat{w}_0) \Big) \Big] - \varepsilon - (2j+2)\delta;\end{aligned} \tag{13.116}$$

(c) Obtain (13.114) from (13.115)–(13.116).

To prove (a), let $\sigma > 0$. Since $U_{1,k} \le u_k \le U_{2,k}$, by (9.110) and (9.115), for $m_{k,2} - m_{k,1} \ge M_0(\sigma)$ (or equivalently $k \ge k_3(\sigma)$), $r \ge r_0(\sigma)$, and any $j \in \mathbb{Z}$,

$$\|u_k - U_{1,k}\|_{L^2(S_j \cap [(-\infty < x_1 \le m_{k,1} - r) \cup (m_{2,k} + r \le x_1 < \infty)])} \le \kappa_2(\sigma) \tag{13.117}$$

and

$$\|u_k - U_{1,k}\|_{L^2(S_j \cap (m_1 + r \le x_1 \le m_2 - r))} \le \kappa_5(\sigma). \tag{13.118}$$

For $|j| \ge j_1$, u_k is a solution of (PDE) on S_j. Hence using (PDE) as in earlier sections, (13.117) and (13.118), for $|j| \ge j_1$ and $k \ge k_3(\sigma)$, we have,

$$\|u_k - U_{1,k}\|_{W^{1,2}(S_j \cap D_k)} \le \kappa_6(\sigma) \tag{13.119}$$

where

$$D_k = (-\infty < x_1 \le m_{1,k} - r] \cup [m_{1,k} + r \le x_1 \le m_{2k} - r] \cup [m_{2,k} + r \le x_1 < \infty).$$

By Proposition 4.16 and (PDE), interpolation estimates show that $\tau^2_{-j}\bar{u}_1 \to v_1$ as $j \to \infty$ uniformly on $S_0 \cap [-r \le x_1 \le r]$. Take $r = r_0(\sigma)$. Then there is a $j_3 = j_3(\sigma) > j_1$ such that for $x_2 \le -j_3$,

$$\int_{-r}^{r} |\bar{u}_1 - v_1|^2 dx_1 \le \sigma. \tag{13.120}$$

Since $\tau^1_{-m_{k,1}} u_k \to \bar{u}_1$ in L^2_{loc} along a subsequence as $k \to \infty$ and therefore uniformly on compact sets, due to standard PDE estimates,

$$\int_{-r}^{r} |\tau^1_{-m_{k,1}} u_k - \bar{u}_1|^2 dx \le \sigma \tag{13.121}$$

for $-j - 1 \le x_2 \le -j$ and $k \ge k_4(\sigma, j)$. Similarly, with the aid of Theorem 9.9,

$$\int_{-r}^{r} |\tau^1_{m_{k,1}} U_{1,k} - v_1|^2 dx_1 \le \sigma \tag{13.122}$$

for $-j - 1 \le x_2 \le -j$ and $k \ge k_4(\sigma, j)$. Combining (13.120)–(13.122) shows that

$$\int_{m_{k,1}-r}^{m_{k,1}+r} |u_k - U_{1,k}|^2 dx \le \kappa_7(\sigma) \tag{13.123}$$

for the above j, k. Thus (13.123) with its analogue for $m_{k,2}$ and (13.119) yield

$$\|u_k - U_{1,k}\|_{L^2(S_{-j})} \le \kappa_8(\sigma), \tag{13.124}$$

and again as in earlier sections,

$$\|u_k - U_{1,k}\|_{W^{1,2}(S_{-j})} \le \kappa_9(\sigma) \tag{13.125}$$

for $j \ge j_3(\sigma)$ and $k \ge k_4(\sigma, j)$.

For $i \gg j$, define

$$\chi_{i,j,k} = \begin{cases} U_{1,k}, & -i-1 \le x_2 \le -i, \\ u_k, & -i+1 \le x_2 \le -j, \\ U_{1,k}, & -j+1 \le x_2 \le -j+2, \end{cases} \tag{13.126}$$

with the usual interpolation and extend $\chi_{i,j,k}$ to $\mathbb{R}$ as an $(i-j+2)$ periodic function of x_2. Proposition 13.22 implies

$$\hat{J}_2(\chi_{i,j,k}) \ge 0. \tag{13.127}$$

Now to get (13.115), write

$$\sum_{t\le 0} \hat{J}_{2,t}(u_k) = \hat{J}_{2;-\infty,-i}(u_k) + \hat{J}_2(\chi_{i,j,k}) - \hat{J}_{2,-i}(\chi_{i,j,k}) - \hat{J}_{2,-j}(\chi_{i,j,k}) + \hat{J}_{2,-j,0}(u_k). \tag{13.128}$$

For $j \ge j_3$, by (13.125),

$$|\hat{J}_{2,-i}(\chi_{i,j,k})|, |\hat{J}_{2,-j}(\chi_{i,j,k})| \le \kappa_{10}(\sigma), \tag{13.129}$$

and for $i = i(k, \sigma)$ large, by Proposition 13.30,

$$|\hat{J}_{2;-\infty,i}(u_k)| \le \kappa_{10}(\sigma). \tag{13.130}$$

Therefore by (13.128)–(13.130),

$$\sum_{t\le 0} \hat{J}_{2,t}(u_k) \ge \sum_{-j}^{0} \hat{J}_{2,t}(u_k) - 3\kappa_{10}(\sigma). \tag{13.131}$$

Getting a similar estimate for positive indices and then choosing $\sigma = \sigma(\varepsilon)$ small enough yields (13.115).

To prove (B), for $|i| \le j$, write

$$\begin{aligned}\hat{J}_{2,i}(u_k) = J_1(\tau^2_{-i}u_k) - b_1 &= J_{1;-\infty,m_1-R-1}(\tau^2_{-i}u_k) + J_{1;-R,R}(\tau^2_{-i}\tau^1_{-m_{k,1}}u_k) \\ &+ J_{1;m_1+R,m_2-R+1}(\tau^2_{-i}u_k) + J_{1;-R,R}(\tau^2_{-i}\tau^1_{-m_{k,2}}u_k) \\ &+ J_{1;m_2+R,\infty}(\tau^2_{-i}u_k) - b_1.\end{aligned} \tag{13.132}$$

As in (9.38) (with $\tilde{c} = 0$), for $k \geq k_5(\delta)$,

$$b_1 \leq c_1(v_0, w_0) + c_1(\hat{v}_0, \hat{w}_0) + \delta. \tag{13.133}$$

We claim that for $R \geq R_1(s)$ and $k \geq k_6(s)$,

$$|J_{1;-\infty,m_1-R-1}(\tau^2_{-i}u_k)|, |J_{1;m_1+R,\infty}(\tau^2_{-i}u_k)|, |J_{1;m_1+R,m_2-R-1}(\tau^2_{-i}u_k)| \leq \kappa_{11}(s). \tag{13.134}$$

Assuming (13.134) for the moment, by (13.115) and (13.132)–(13.134),

$$\sum_{|i|\leq j} J_{2,i}(u_k) \geq \sum_{|l|\leq j} \Big[J_{1;-R,R}(\tau^2_{-i}\tau^1_{-m_{k,1}}u_k) - c_1(v_0, w_0) + J_{1;-R,R}(\tau^2_{-i}\tau^1_{-m_{k,2}}u_k) - c_1(\hat{v}_0, \hat{w}_0) \Big] - \varepsilon - (2j+1)3\kappa_{11}(s) - \delta. \tag{13.135}$$

Thus choosing s so small that $3\kappa_{11}(s) \leq \delta$ yields (13.116).

Now to prove (13.134), we argue as in Proposition 9.107. By (9.118),

$$\int_{-\infty}^{m_{k,1}-R} (\tau^2_{-i}u_k - U_{1,k})dx_1 \leq \int_{-\infty}^{m_{k,1}-R} (\tau^1_{-1}U_{1,k} - U_{1,k})dx_1 \leq \int_{m_{k,1}-R}^{m_{k,1}-R+1} (U_{1,k} - v_0)dx_1, \tag{13.136}$$

so (9.108) implies

$$\int_{S_{0;-\infty,m_1-R}} (\tau^2_{-r}u_k - U_1)dx \leq \|U_1 - v_0\|_{L^2(T_{m_1-R})} \leq s. \tag{13.137}$$

Since $\tau^2_{-r}u_k$ satisfies (PDE) in $S_{0;-\infty,m_{k,1}-R}$ for $R > l_0$,

$$-\Delta(\tau^2_{-i}u_k - U_{1,k}) + (F_u(x, \tau_{-i}u_k) - F_u(x, U_{1,k})) = 0 \tag{13.138}$$

in that region. Multiplying (13.138) by $\tau^2_{-i}u_k - U_{1,k}$, integrating over $S_{0;p,m_{k,1}-R}$, and letting $p \to -\infty$ gives

$$\int_{S_{0;-\infty,m_{k,1}-R}} |\nabla(\tau^2_{-i}u_k - U_{1,k})|^2 dx = \int_{\partial S_{0;-\infty,m_{k,1}-R}} (\tau^2_{-r}u_n - U_{1,k})\frac{\partial}{\partial\nu}(\tau^2_{-i}u_k - U_{1,k})dH^{n-1} - \int_{S_{0;-\infty,m_{k,1}-R}} (F_u(x, \tau^2_{-i}u_k) - F_u(x, U_{1,k}))dx. \tag{13.139}$$

Estimating the boundary terms as in earlier cases with the aid of (13.136) yields

$$\|\tau^2_{-i}u_k - U_{1,k}\|_{L^2(S_{0;-\infty,m_{k,1}-R})} \leq \kappa_{12}(s). \tag{13.140}$$

This with (9.116) for $u = \tau^2_{-i}u_k$ and (9.109) establishes (13.134) for $J_{1;-\infty,m_{k,1}-R-1}(\tau^2_{-i}u_k)$. A similar argument gives the estimate for $J_{1;m_{k,1}+R,\infty}(\tau^2_{-i}u_k)$. Lastly, to get estimates for the intermediate region, since $U_{1,k} \leq u_k \leq U_{2,k} \leq \tau^1_{-1}U_{1,k}$,

$$\int_{m_{k,1}+R}^{m_{k,2}-R} (\tau^2_{-i}u_k - U_{1,k})dx_1 \leq \int_{m_{k,1}+R}^{m_{k,2}-R} (\tau^1_{-i}U_{1,k} - U_{1,k})dx_1$$
$$= -\int_{m_{k,1}+R}^{m_{k,1}+R+1} (U_{1,k} - w_0)dx_1 + \int_{m_{k,2}-R}^{m_{k,2}-R+1} (U_{1,k} - w_0)dx_1. \tag{13.141}$$

Thus by (13.141) and (9.113), as in (13.137) we have

$$\int_{S_{0;m_{k,1}+R,m_{k,2}-R-1}} (\tau^2_{-i}u_k - U_{1,k})dx \leq \|U_{1,k} - w_0\|_{L^2(T_{m_{k,1}}+R)}$$
$$+\|U_{1,k} - w_0\|_{L^2(T_{m_{k,2}-R})} \leq 2\kappa_3(s). \tag{13.142}$$

We can assume that k is so large that $\tau^2_{-r}u_k$ satisfies (PDE) in $S_{0;m_{k,1}+R,m_{k,2}-R}$. Therefore following (13.138)–(13.140) and using (9.114) then gives (13.134) for $J_{1;m_{k,1}+R,m_{k,2}-R}(\tau^2_{-r}u_k)$.

Next to prove (13.134), by (13.140), for $k \geq k_6(\delta)$,

$$c_2(v_1, w_1) + c_2(\hat{v}_1, \hat{w}_1) + \delta \geq \sum_{|i|\leq j} \Big[(J_{1;-R,R})\left(\tau^2_{-i}\tau^1_{-m_{k,1}}u_k\right) - c_1(v_0, w_0)$$
$$+ \left(J_{1;-R,R}\left(\tau^2_{-i}\tau^1_{-m_{k,2}}u_k\right) - c_1\left(\hat{v}_0, \hat{w}_0\right)\right)\Big]$$
$$-\varepsilon - (2j+2)\delta. \tag{13.143}$$

Choose δ such that $(2j+3)\delta = \epsilon$, so (13.143) becomes

$$c_2(v_1, w_1) + c_2(\hat{v}_1, \hat{w}_1) \geq \sum_{|i|\leq j} \Big[\Big(J_{1;-R,R}(\tau^2_{-i}\tau^1_{-m_{k,1}}u_k) - c_1(v_0, w_0)\Big)$$
$$+ \left(J_{1;-R,R}(\tau^2_{-i}\tau^1_{m_{k,2}}u_k) - c_1(\hat{v}_0, \hat{w}_0)\right)\Big] - 2\varepsilon. \tag{13.144}$$

Letting $k \to \infty$ and using the weak lower semicontinuity of $J_{1;p,q}$ gives

$$c_2(v_1, w_1) + c_2(\hat{v}_1, \hat{w}_1) \geq \sum_{|i|\leq j} \big[J_{1;-R,R}(\tau^2_{-i}\bar{u}_1) - c_1(v_0, w_0) + J_{1;-R,R}(\tau^2_{-i}\bar{u}_2) - c_1(\hat{v}_0, \hat{w}_0)\big] - 2\varepsilon. \tag{13.145}$$

Now let $R \to \infty$, then $j \to \infty$, and finally $\varepsilon \to 0$, yielding (13.114) and 2^o(a) of Theorem 13.38.

To complete the proof of Theorem 13.38, note that 2^o(b) follows as usual from 2^o(a) as in earlier arguments. Likewise, 2^o(c), (3^o) follow from 2^o(a) and standard maximum principle arguments exploiting the fact that all constraints in the definition of $\mathcal{Y}_m$ are pointwise constraints. The proof of Theorem 13.38 is complete.

Remark 13.146. Another class of solutions of (PDE) on $\mathbb{R}^2 \times \mathbb{T}^{n-2}$ that it is natural to seek is the class of solutions that approach two transition solutions as obtained in Theorem 6.8 corresponding to different values of m as $x_2 \to \pm\infty$. Whether such solutions exist remains an open question. Likewise, based in the results of Chapters 6–11, there is a rich variety of other possible x_2 asymptotics for solutions that one could pursue. Existence for all of these cases remains unknown.

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